D

Mathematical Reviews

Edited by

W. Feller **E. Hille** **H. Whitney**

R. P. Boas, Jr., ***Executive Editor***

Vol. 11, No. 6 June, 1950 pp. 411–486

TABLE OF CONTENTS

AUTHOR INDEX

(*Continued on cover 4*)

ONE-SIDE EDITION OF MATHEMATICAL REVIEWS

An edition of MATHEMATICAL REVIEWS printed on only one side of the paper is available to persons interested in making card files of reviews or wishing to add remarks to reviews in the future. This special edition may be obtained for an additional payment of $1.00. A regular current subscription can be changed to a one-side subscription by paying the additional $1.00. This edition is folded but not stitched.

MATHEMATICAL REVIEWS

Published monthly, except August, by

THE AMERICAN MATHEMATICAL SOCIETY, Prince and Lemon Streets, Lancaster, Pennsylvania

Editorial Office

MATHEMATICAL REVIEWS, Brown University, Providence 12, R. I.

Subscriptions: Price $20 per year ($10 per year to members of sponsoring societies). Checks should be made payable to MATHEMATICAL REVIEWS. Subscriptions may be addressed to MATHEMATICAL REVIEWS, Lancaster, Pennsylvania, or Brown University, Providence 12, Rhode Island, but should preferably be addressed to the American Mathematical Society, 531 West 116th Street, New York 27, N. Y.

This publication was made possible in part by funds granted by the Carnegie Corporation of New York, the Rockefeller Foundation, and the American Philosophical Society held at Philadelphia for Promoting Useful Knowledge. Its preparation is also supported currently under a contract with the Office of Air Research, Department of the Air Force, U. S. A. These organizations are not, however, the authors, owners, publishers, or proprietors of this publication, and are not to be understood as approving by virtue of their grants any of the statements made or views expressed therein.

Entered as second-class matter February 3, 1940 at the post office at Lancaster, Pennsylvania, under the act of March 3, 1879. Accepted for mailing at special rate of postage provided for in paragraph (d-2), section 34.40, P. L. and R. of 1948, authorized November 9, 1940.

Mathematical Reviews

Vol. 11, No. 6 **JUNE, 1950** **Pages 411-486**

FOUNDATIONS

***Hilbert, D., und Ackermann, W. Grundzüge der theoretischen Logik.** Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen mit besonderer Berücksichtigung der Anwendungsgebiete, Band XXVII. 3d ed. Springer-Verlag, Berlin-Göttingen-Heidelberg, 1949. viii+155 pp.

The first edition of this excellent textbook appeared in 1928, the second in 1938. The present third edition involves comparatively minor changes of which the most important is a fuller treatment of the "Stufenkalkül." For further details see the review by Church [J. Symbolic Logic **15**, 59 (1950)]. *H. B. Curry* (State College, Pa.).

Wang, Hao. A proof of independence. Amer. Math. Monthly **57**, 99–100 (1950).

In the first volume of "Grundlagen der Mathematik," by Hilbert and Bernays [Springer, Berlin, 1934], a system of fifteen axioms for the propositional calculus is given, and the independence of the first axiom is proved by means of truth tables involving four truth values. The author of this paper proves the same thing using only three truth values. *O. Frink* (State College, Pa.).

Halldén, Sören. On the decision-problem of Lewis' calculus S5. Norsk Mat. Tidsskr. **31**, 89–94 (1949).

A formula P of the Lewis algebra S5 [C. I. Lewis and C. H. Langford, Symbolic Logic, Century, New York, 1932, p. 501] is called an F-formula if and only if P contains $\Diamond$, and is such that if it contains a conjunction $R.S$, R and S both contain $\Diamond$. The author gives a simple matrix M and proves that an F-formula P is a theorem of S5 if and only if it satisfies M. This M is 3-valued with 1 as designated value and $\sim 1=2$, $\sim 2=1$, $\sim 3=3$, $\Diamond 1=\Diamond 3=1$, $\Diamond 2=2$, $1.1=1$, $1.2=2.1=2.2=2$. (The conjunctive values where 3 is involved are irrelevant.) Although general decision methods for S5 are known, this special case has some interest because of its simplicity. The author then applies this result to derive an analogue of a theorem of McKinsey and Tarski [J. Symbolic Logic **13**, 1–15 (1948); these Rev. **9**, 486] concerning certain translations T_1, T_2, T_3 from the Heyting algebra to S4. It is shown that P is a theorem of the classical propositional algebra if and only if $T_1(p)$ and $T_3(p)$ are theorems of S5. [On p. 91 Q and R in several places should be Q' and R'.] *H. B. Curry.*

Kalmár, László. Une forme du théorème de Gödel sous des hypothèses minimales. C. R. Acad. Sci. Paris **229**, 963–965 (1949).

The author proves an abstract form of the Gödel incompleteness theorem, generalizing still further the work of Chauvin [same C. R. **228**, 1085–1087, 1179–1180 (1949); these Rev. **10**, 668]. Kalmár's theorem can be stated very compactly if we let "FAB" designate, for given classes A and B, the class of functions on A to B, as follows. Let Θ be a theory consisting of two classes P (propositions) and D (demonstrations) and a function $\kappa \varepsilon FDP$. Let N be the class of natural (nonnegative) integers and Φ a subclass of FNN; then we suppose Θ such that there exists a $\lambda \varepsilon F\Phi(FNP)$, where we think of $\lambda(\phi, n)$, for $\phi \varepsilon \Phi$ and $n \varepsilon N$, as meaning "$\phi(m) \neq n$ for all $m \varepsilon N$." We further postulate two functions $\psi \varepsilon F\Phi N'$ and $\chi \varepsilon FDN'$, where $N'=N-\{0\}$; these are assumed one-to-one, so that they have inverses $\psi^{-1} \varepsilon FN''\Phi$ and $\chi^{-1} \varepsilon FN'''D$, where N'' and N''' are subclasses of N'. We then define $\gamma \varepsilon FNN$ thus: if $k=\chi(d)$, where $\kappa(d)$ is $\lambda(\psi^{-1}(n), n)$, then $\gamma(k)$ is n; otherwise $\gamma(k)$ is 0. If $\gamma \varepsilon \Phi$, Θ is called Gödelian. In that case let $l=\psi(\gamma)$, $p=\lambda(\gamma, l)$. If $p=\kappa(d)$, then $\gamma(k)=l$ for $k=\chi(d)$; in that case we say that p (and Θ) are incorrect. If p is not a theorem of Θ, then p is true in interpretation but unprovable, and Θ is incomplete. In this sense every Gödelian Θ is either incorrect or incomplete. *H. B. Curry* (State College, Pa.).

Kalmár, László. Quelques formes générales du théorème de Gödel. C. R. Acad. Sci. Paris **229**, 1047–1049 (1949).

This is a continuation of the paper reviewed above. The author shows that if additional hypotheses are introduced, then one can deduce the Gödel theorem in more familiar forms. E.g., if one postulates a $\mu \varepsilon FN(FNP)$, one can think of $\mu(m, n)$ as representing $m=n$, and define inconsistency and ω-noncategoricalness accordingly; then a Gödelian Θ with such a μ is either inconsistent or ω-noncategorical. Further forms of the theorem are obtained when negation is postulated. *H. B. Curry* (State College, Pa.).

Carruccio, Ettore. Sulla potenza dell'insieme delle proposizioni di un dato sistema ipotetico-deduttivo. Boll. Un. Mat. Ital. (3) **4**, 299–306 (1949).

The author gives conditions under which it can be proved that the set of all propositions of a formal deductive system is not only denumerable but effectively denumerable, and states that these conditions hold for the system of Hilbert. *O. Frink* (State College, Pa.).

Lorenzen, Paul. Einführung in die Logik. Arch. Math. **2**, 60–65 (1949).

The author sketches a formal system for the arithmetic of natural numbers. Besides deduction rules like $\mathfrak{a}, \mathfrak{b} \blacktriangleright \mathfrak{a} \wedge \mathfrak{b}$ (if a and b are deducible formulas, then $a \wedge b$ is a deducible formula) there are metarules, e.g., the induction rule

$$\frac{\mathfrak{a} \blacktriangleright \mathfrak{c}(|) \quad \mathfrak{a}, \mathfrak{c}(b) \blacktriangleright \mathfrak{c}(b|)}{\mathfrak{a} \blacktriangleright \mathfrak{c}(b)},$$

meaning: if by the substitution of certain formulas for $\mathfrak{a}$, $\mathfrak{c}$ the rules above the bar become valid deduction rules, then the rule below the bar becomes valid as well. In this system the rule $\gamma \blacktriangleright \mathfrak{c} \vee \neg \mathfrak{c}(T)$ is not provable (γ means truth, $\neg$ negation), but it can be shown that by the adjunction of (T) no disjunction-free formula becomes deducible that was not deducible before. No proofs are given. It is indicated that the method applies also to the foundations of set theory and analysis. *A. Heyting* (Amsterdam).

Sobociński, Bolesław. L'analyse de l'antinomie russellienne par Leśniewsi. II. Methodos 1, 220–228 (1949).

In continuation of a previous paper [same vol., 94–107 (1949); these Rev. 11, 73], the author shows that a contradiction can still be deduced if one of the assumptions is weakened in a way suggested by Frege. A third paper on the same subject is to follow. *H. B. Curry.*

van Dantzig, D. Comments on Brouwer's theorem on essentially-negative predicates. Nederl. Akad. Wetensch., Proc. **52**, 949–957 = Indagationes Math. **11**, 347–355 (1949).

The author tries to clarify Brouwer's note [same Proc. **51**, 963–964 = Indagationes Math. **10**, 322–323 (1948); these Rev. **10**, 421] by using a more "objectivistic" terminology. Let $\omega_1, \omega_2, \cdots$ be finite sets of mathematical deductions and A a mathematical assertion. If $\sigma_n = \bigcup_1^n \omega_i$ contains neither a deduction of A nor a deduction of the absurdity of A, then $a_n = 0$; if r is the first number such that σ_r contains a deduction of A (or $\neg A$, respectively), then $a_m = 2^{-r}$ (or -2^{-r}, respectively) for every $m \geqq r$. Then $\lim_{n\to\infty} a_n = \rho$. Now it is absurd that, for every choice of $\omega_1, \omega_2, \cdots, \rho = 0$. A still more strict formulation, referring to a definite formal system, is given. *A. Heyting* (Amsterdam).

Billing, J. The principle of the excluded third and the Bolzano-Weierstrass lemma. Ark. Mat. **1**, 59 (1949).

The author interprets the law of excluded middle as saying roughly that every proposition is either true or false provided "that our knowledge suffices to decide the truth-value of the proposition." According to the author, the classical proof of the Bolzano-Weierstrass lemma, utilizing the law of excluded middle without such a proviso, breaks down. The proof must rather utilize the law of excluded middle with the proviso attached. Precisely what the added proviso means and how it would be formulated in a given logistic system are not discussed. *R. M. Martin.*

Kurepa, Georges. Démonstration du principe de l'induction totale. C. R. Acad. Sci. Paris **230**, 703–705 (1950).

This is a proof of the validity of mathematical induction for the natural numbers. It is difficult to understand the significance of the paper, however, since the author never explains exactly what principles of set theory he is assuming. There are gaps in the argument, moreover, which the reviewer does not see offhand how to bridge: in the proof of 5°, for instance, it would apparently be necessary, before one could apply 2°, to show that the set of all numbers m_n, for $n \in N_0$, is itself a subset of N_0. *J. C. C. McKinsey.*

ALGEBRA

Frankel, E. T. A calculus of figurate numbers and finite differences. Amer. Math. Monthly **57**, 14–25 (1950).

This paper has to do with sets of numbers defined by the following formulas where standard combinatorial notation is used:

$$F_r^n = \binom{n+r-1}{r}, \quad F_r^{-n} = (-1)^r \binom{n}{r}.$$

The numbers in one guise or another have, of course, been studied for a long time and naturally many of the relations which the author develops are already known or readily derived from relations which are known. The two principal ideas of the paper are (1) the "inverse of a sum" and (2) "criss-cross" multiplication. In most works on finite differences the operator Δ is defined and then an operator Σ, known as the indefinite sum operator, inverse to Δ, is defined. The "definite sum" then appears as a particular determination of the indefinite sum. The reviewer regards the term "inverse of a sum" as an unwarranted departure from customary usage. For example, if in the author's table 1 a first row of 0's is introduced, there seems no reason not to say that each column is obtained by differencing the column which succeeds it as well as by summing the column which precedes it. The term "criss-cross" multiplication of two series is used to describe what in the theory of infinite series is usually known as the Cauchy product. The author uses "criss-cross" multiplication to obtain certain results which apparently can be readily obtained also by the summation by parts formula,

$$\sum_{i=M}^{n} s_i \epsilon_i = \epsilon_{n+1} \sum_{i=M}^{n} s_i - \sum_{i=M}^{n} \Delta\epsilon_i \sum_{j=M}^{i} s_j.$$

T. Fort (Athens, Ga.).

Sibirani, Filippo. Di alcune identità. Boll. Accad. Gioenia Sci. Nat. Catania (4) **2**, 119–125 (1949).

Some identities are obtained from binomial expansions by differentiating and integrating with respect to parameters. *O. Todd-Taussky* (Washington, D. C.).

Verdenius, W. On the number of terms of the square and the cube of polynomials. Nederl. Akad. Wetensch., Proc. **52**, 1220–1226 = Indagationes Math. **11**, 459–465 (1949).

The author proves that for any n there exists a real polynomial $f(x) = \sum_{k=0}^n a_k x^k$, $a_k \neq 0$, so that $f^2(x)$ has less than cn^b terms, $b = \log_{12} 8$. This sharpens a result of the reviewer [Nieuw Arch. Wiskunde **23**, 63–65 (1948); these Rev. **10**, 354]. He also constructs a real polynomial of the same kind, such that $f^3(x)$ has less than cn^γ terms, where $\gamma < 1$ is independent of n. The statement of the first result in the introduction contains a misprint. *P. Erdös.*

★Mal'cev, A. I. Osnovy Lineĭnoĭ Algebry. [**Foundations of Linear Algebra**]. OGIZ, Moscow-Leningrad, 1948. 423 pp.

Matrices, Linear spaces, Linear transformations, Jordan's normal form, Polynomial matrices, Matrices (concluded), Unitary and Euclidean spaces, Symmetric transformations, Spaces with bilinear metric, Linear transformations of bilinear-metric spaces. *Table of contents.*

Parker, W. V. The matrix equation $AX = XB$. Duke Math. J. **17**, 43–51 (1950).

A parametric solution for the matrix equation $AX = XB$, where A and B are n by n and m by m, respectively, is found by taking A in rational canonical form. Thus A is written as a direct sum of nonderogatory matrices A_i ($i = 1, \cdots, k$) with characteristic functions $\Phi_i(\lambda)$, where Φ_i is divisible by Φ_{i+1} and the Φ_i (of degree n_i) are the invariant factors of $\lambda I - A$. Then $AX = XB$ if and only if X', the transpose of X, has the form $\|X_1', \cdots, X_k'\|$, where X_i has the successive rows $x_i, x_iB, \cdots, x_iB^{n_i-1}$ and x_i is any m-component vector such that $x_i\Phi_i(B) = 0$. The method of the paper is suggested by the statement of this theorem. There is a systematic discussion, with two illustrative examples, which includes such results as: the only matrices commutative with a nonderogatory matrix A are polynomials in A; the number of linearly independent matrices commuta-

tive with A is $\sum_{i=1}^{k}(2i-1)n_i$; and, if X commutes with every matrix commutative with A, then X is a polynomial in A. In terms of the explicit parametrization of an X commutative with A, the characteristic equation of X is found and a polynomial $F(\mu, \lambda)$ is obtained such that $F(X, A)=0$.

The emphasis in the paper is on the explicit character of the solutions obtained and on the elementary character of the methods used. The author [in a conversation] confirms the reviewer's opinion that most of the results were known previously. There is no reference to the paper by Ingraham and Trimble [Amer. J. Math. **63**, 9–28 (1941); these Rev. **2**, 243] where the more general equation $AX=XB+C$ is treated without the requirement that A be in rational canonical form. *W. Givens* (Knoxville, Tenn.).

Parodi, Maurice. Sur une limite supérieure du rapport des valeurs caractéristiques de deux matrices symétriques, définies positives, à éléments réels, dont les éléments correspondants diffèrent peu. C. R. Acad. Sci. Paris **230**, 705–707 (1950).

Upper bounds for roots of Lagrangian matrices $(A\mu+B)$ are studied. One method is to consider the matrix $(\mu+A^{-1}B)$. Since, however, inversion of matrices is not always practicable an alternative method is suggested.
O. Todd-Taussky (Washington, D. C.).

Garnir, H. Sur les systèmes de matrices hermitiennes $A_1, \cdots, A_n$ vérifiant les relations

$$A_iA_jA_k+A_kA_jA_i=A_i\delta_{jk}+A_k\delta_{ji},$$

$(i, j, k=1, \ldots, n)$. Acad. Roy. Belgique. Cl. Sci. Mém. Coll. in 8°. (2) **23**, no. 9, 28 pp. (1949).

The essential content of this paper has been published by N. Kemmer [Proc. Cambridge Philos. Soc. **39**, 189–196 (1943); these Rev. **5**, 225], D. E. Littlewood [ibid. **43**, 406–413 (1947); these Rev. **9**, 171] and N. Svartholm [Kungl. Fysiografiska Sällskapets i Lund Förhandlingar [Proc. Roy. Physiog. Soc. Lund] **12**, no. 9, 94–108 (1942); these Rev. **7**, 4]. The author states that Kemmer's paper was unavailable to him when the work was done and Littlewood's paper appeared after the one under review had been accepted for publication [1946]. Svartholm's work has been overlooked by the later authors. Irreducible matrices satisfying the equation in the title are constructed from linear operators on certain skew-symmetric tensors. Known numerical examples are given explicitly for $n=4$ and 5. It is proved that all inequivalent representations have been obtained. [Of eight references, three list pagination incorrectly, three are incomplete and one (Littlewood, cited above) is incorrect as to volume and year. There are several minor errors and misprints and the notation for conjugate (star), transpose (tilde) and conjugate transpose (plus) is not explained.] *W. Givens* (Knoxville, Tenn.).

***Gurevič, G. B. Osnovy Teorii Algebraičeskih Invariantov. [Foundations of the Theory of Algebraic Invariants].** OGIZ, Moscow-Leningrad, 1948. 408 pp.

Geometrical introduction; Foundations of tensor algebra; Invariants and comitants of tensors and their simplest properties; Fundamental theorem of the theory of invariants and its consequences; Binary forms; Ternary forms, tensors of valence two; Polyvectors. *Table of contents.*

Strom, Carl W. Complete systems of invariants of the cyclic groups of equal order and degree. Proc. Iowa Acad. Sci. **55**, 287–290 (1948).

The invariants are discussed of functions of $x_0, x_1, x_2, \cdots, x_{n-1}$ under the cyclic group of permutations. Putting $y_i=x_0+\epsilon^i x_1+\epsilon^{2i}x_2+\cdots+\epsilon^{(n-1)i}x_i$, where $\epsilon=e^{2\pi i/n}$, then under a cyclic permutation y_i becomes $\epsilon^i y_i$. A polynomial in the x_i is expressible as a polynomial in the y_i. The product $y_0^{\alpha_0}y_1^{\alpha_1}\cdots y_{n-1}^{\alpha_{n-1}}$ is invariant if $\alpha_1+2\alpha_2+\cdots+(n-1)\alpha_{n-1}\equiv 0 \pmod{n}$, and the irreducible system is obtained from the solutions of this congruence in positive integers not expressible as a sum of other similar solutions. No general solution is possible, but the calculation is greatly simplified by considering conjugate sets of solutions. Given one solution other solutions can be obtained from it by considering the automorphisms of the cyclic group, and the whole set of solutions obtained in this way is a complete set of conjugate solutions. If one solution of the set is included in the irreducible system then the whole conjugate set must be so included. An analysis is made for $n=1, 2, \cdots, 10$.
D. E. Littlewood (Bangor).

Abstract Algebra

***Moura Mousinho, Maria Laura. Espaços Projetivos. Reticulado de Seus Sub-Espaços. [Projective Spaces. Lattice of Their Sub-Spaces].** Thesis, Rio de Janeiro, 1949. iv+36 pp.

The principal result of this dissertation is the theorem that a necessary and sufficient condition that a lattice be isomorphic to the lattice of all linear subspaces of a projective space of finite or infinite dimension is that it be complete, atomic, complemented and modular, and that it have the "finite dependence property." This latter condition means that every atom of the lattice which is contained in the join of an infinite set A of atoms is contained in the join of a finite subset of A. As the author points out, the necessity, but not the sufficiency of these conditions was proved in a paper of the reviewer [Trans. Amer. Math. Soc. **60**, 452–467 (1946); these Rev. **8**, 309]. *O. Frink.*

Halanay, A. Théorèmes de Jordan-Hölder dans la théorie des structures. Disquisit. Math. Phys. **7**, 3–23 (1948).

The author examines the concepts of normality in lattices due to Barbilian, Kurosch and the reviewer, and shows that they are in fact the same. This makes it possible to supplement a refinement theorem by the reviewer on normal chains in lattices and obtain a close analogue to the group theorem of Jordan-Hölder. These questions are also studied especially in metric lattices and a characterisation of the normal elements is given. Finally it is shown that in p-groups certain weak forms of normality imply the stronger.
O. Ore (New Haven, Conn.).

Cohen, I. S. Commutative rings with restricted minimum condition. Duke Math. J. **17**, 27–42 (1950).

The rings which are dealt with in this paper are commutative and have an identity element; an ideal in a ring R is called proper if it is different from both $\{0\}$ and R. The object of the paper is to study the interrelationship of various conditions which a ring may satisfy: the maximum condition (Noetherian rings), the minimum condition, the restricted minimum condition (i.e., the minimum condition

should hold in $R/\mathfrak{a}$ for every proper ideal $\mathfrak{a}$), the condition of being a Dedekind ring, the condition of being of finite rank (R is of rank k if every ideal has a set of k generators).

Theorem 1 states that the restricted minimum condition is satisfied in R if and only if R is Noetherian and every proper prime ideal is maximal; this contains the result due to Akizuki [Proc. Phys.-Math. Soc. Japan (3) 17, 337–345 (1935)] to the effect that the restricted minimum condition implies the maximal condition.

Theorem 3 is as follows: let R be a restricted minimum integral domain; let S be an integral domain containing R whose quotient field is finite over that of R and which is integral over R; then S is a restricted minimum ring, and, more precisely, if $\mathfrak{A}$ is a proper ideal of S, then $S/\mathfrak{A}$ has a composition series when considered as an R-module. Various special cases of this result have been proved before; a result whose equivalence to theorem 3 is established in the present paper has been announced without proof by H. Grell [Ber. Math.-Tagung Tübingen 1946, p. 67 (1947); these Rev. 9, 5].

A simple proof is given of the theorem due to Matusita [Jap. J. Math. 19, 97–110 (1944); these Rev. 7, 360] to the effect that an integral domain in which every ideal is a product of prime ideals is a Dedekind ring. The following characterisations of Dedekind rings are given: if R is a Noetherian integral domain, then the following conditions are all equivalent: (1) R is a Dedekind ring; (2) for every maximal ideal $\mathfrak{p}$, the quotient ring $R_\mathfrak{p}$ is a discrete valuation ring; (3) if $\mathfrak{p}$ is a maximal ideal of R, there is no ideal between $\mathfrak{p}$ and $\mathfrak{p}^2$; (4) a primary ideal belonging to a maximal ideal is a product of prime ideals; (5) the set of primary ideals belonging to a maximal ideal is totally ordered; (6) for any three ideals in R, $\mathfrak{a}\cap(\mathfrak{b}+\mathfrak{c})=(\mathfrak{a}\cap\mathfrak{b})+(\mathfrak{a}\cap\mathfrak{c})$; (7) for any three ideals of R, $\mathfrak{a}:(\mathfrak{b}\cap\mathfrak{c})=(\mathfrak{a}:\mathfrak{b})+(\mathfrak{a}:\mathfrak{c})$.

It is proved that a local domain (Noetherian integral domain in which the non-units form an ideal) is of finite rank if and only if it satisfies the restricted minimum condition [the sufficiency had been proved by Akizuki, Jap. J. Math. 14, 85–102 (1938)]. Moreover it is established that for a Noetherian domain R to be of finite rank, it is necessary and sufficient that the ranks of the rings $R_\mathfrak{p}$ for all prime ideals $\mathfrak{p}$ be bounded; if k is an upper bound for these ranks, then R is of rank $k+1$. Some applications are given to modules of finite rank over a ring of finite rank.

C. Chevalley (New York, N. Y.).

Azumaya, Gorô. Galois theory for uni-serial rings. J. Math. Soc. Japan 1, 130–146 (1949).

A ring R, with radical C, is uniserial [Köthe, Math. Z. 39, 31–44 (1934)] if it satisfies the minimal condition, $\bar{R}=R/C$ is simple, and $C=cR=Rc$ for some element c. For a certain class of finite groups $\mathfrak{G}$ of automorphisms of R, the author proves, in much the same way as Nakayama and Azumaya [Ann. of Math. (2) 48, 949–965 (1947); these Rev. 9, 563] do it for irreducible rings, that there is a one-to-one correspondence between subgroups of $\mathfrak{G}$ and uniserial subrings over which R is regular. The condition on $\mathfrak{G}$ is that, if C/C^2 is defined as $\bar{R}$-module in the natural way, there is an automorphism φ of $\bar{R}$ such that $\bar{x}c\equiv c\bar{x}^\varphi$ mod C^2 for all $\bar{x}\in\bar{R}$; $\mathfrak{G}$ must contain no element except the identity which induces on $\bar{R}$ an automorphism congruent, mod inner automorphisms of R, to a power φ^i for $i=0, \cdots, l-1$ where $C^l=(0)$. The paper also contains some new results on modules and commuter rings for rings which have unit element and no nontrivial two-sided ideal, without assuming any chain condition.

G. Whaples (Bloomington, Ind.).

Foster, Alfred L. On n-ality theories in rings and their logical algebras, including tri-ality principle in three valued logics. Amer. J. Math. 72, 101–123 (1950).

Si R est un anneau, ρ une permutation de R, on peut définir sur R une nouvelle loi de composition $(x, y)\rightarrow\rho^{-1}(\rho(x)\rho(y))$; si K est un groupe de permutations de R, les lois de composition ainsi définies sur R forment, avec les permutations de K, ce que l'auteur appelle une K-logique de R. Il examine en particulier dans quels cas l'addition sur R peut se définir par combinaison des opérations de la K-logique, et montre que c'est le cas pour les anneaux booléiens, et leurs généralisations, les 3-anneaux, définis par les identités $3x=0$, $x^3=x$.

J. Dieudonné (Nancy).

Peremans, W. A remark on free algebras. Math. Centrum Amsterdam. Rapport **ZW-1949-015**, 3 pp. (1949). (Dutch)

G. Birkhoff [Lattice Theory, Amer. Math. Soc. Colloquium Publ., v. 25, revised ed., New York, 1948, p. vii; these Rev. 10, 673], starting from an arbitrary class $\mathfrak{A}$ of algebras having the same operations, defines a free $\mathfrak{A}$-algebra with n generators as an algebra F (with the given operations) such that every mapping of the generators into an algebra A of $\mathfrak{A}$ is contained in a homomorphism of F into A, and he states existence and uniqueness theorems. The proof of the uniqueness theorem assumes that a mapping of the generators of one free algebra, F_1, onto those of another, F_2, is contained in a homomorphism of F_1 into F_2: this is not justified because F_2 is not necessarily in $\mathfrak{A}$, and an example shews that the uniqueness theorem is untrue. If we postulate that the free algebra should belong to $\mathfrak{A}$ (as a condition on the free algebra—not as a condition on $\mathfrak{A}$) then the existence theorem fails. If $\mathfrak{A}$ is the set of all algebras with given operations and equationally-defined by given identities, as it is in most applications, then the free algebras are in $\mathfrak{A}$ and all is well.

H. A. Thurston (Bristol).

Mal'cev, A. I. On algebras defined by identities. Mat. Sbornik N.S. 26(68), 19–33 (1950). (Russian)

The author is concerned with (possibly nonassociative) algebras which are free subject to given identities. Examples are: free associative algebra, free Lie algebra, free nilpotent algebra of index k. We quote two representative theorems. (1) Over an infinite base field, any such algebra is generalized nilpotent (the intersection of its powers is 0). (2) If A is nilpotent (as well as free subject to identities), and I is a two-sided ideal in A, then any automorphism of A/I is induced by an automorphism of A. Analogous theorems are valid for groups, but in the second, I must be a normal subgroup containing the commutator group. The final two sections deal with the classification of identities. Over an infinite field one can reduce to the case of homogeneous multilinear identities. The author sketches a classification of the latter, from the point of view of the representations of the symmetric group, and gives the details for degrees 2 and 3.

I. Kaplansky (Chicago, Ill.).

Hochschild, G. Note on maximal algebras. Proc. Amer. Math. Soc. 1, 11–14 (1950).

L'auteur précise, dans un cas particulier, la notion d'algèbre maximale, qu'il a introduite dans un travail antérieur [Bull. Amer. Math. Soc. 53, 369–377 (1947); ces Rev. 8, 561]. Il considère une algèbre complètement primaire B sur un corps F, et son radical R; B/R est alors un surcorps non commutatif A de F, et le cas que considère

l'auteur est celui où A est une extension galoisienne de F, au sens de la théorie de Jacobson-Cartan. Il montre alors comment, en supposant connu le groupe des automorphismes extérieurs de A sur F, le plus petit exposant n tel que $R^n=0$, et le nombre s de A-bimodules simples en lequel se décompose le A-bimodule R/R^2, on peut construire une algèbre maximale bien déterminée, dont B est une algèbre quotient.

J. Dieudonné (Nancy).

THEORY OF GROUPS

Schogt, C. Some theorems of Lubelski on group theory. Math. Centrum Amsterdam. Rapport **ZW-1949-012**, 13 pp. (1949). (Dutch)

Starting from the axioms and proving the necessary elementary theorems on the way, the author proves the following theorems. (i) If M is a factor of the order of a finite group and p is a prime factor of M, then M/p is a factor of the number of group elements whose periods are factors of M and multiples of any power of p which is a factor of M. (ii) If $P=\prod_{i=1}^{g} p_i^{b_i}$ and $M=\prod_{i=1}^{h} p_i^{a_i}$, where $0<b_i\leqq a_i$ for i from 1 to g and $h\geqq g$ and the p_i are distinct primes, and if M is a factor of the order of a finite group, then $M/\prod_{i=1}^{g} p_i^{a_i-b_i+1}$ is a factor of the number of group elements whose periods are factors of M and multiples of P.

H. A. Thurston (Bristol).

Staal, R. A. Star diagrams and the symmetric group. Canadian J. Math. **2**, 79–92 (1950).

Nakayama's conjecture [Jap. J. Math. **17**, 165–184, 411–423 (1941); these Rev. **3**, 195, 196; **4**, 340] was proved by Robinson and Brauer [Robinson, Trans. Roy. Soc. Canada. Sect. III. (3) **41**, 11–19 (1947); Brauer, ibid., 20–25 (1947); these Rev. **10**, 678] making use of the formula (A) $e(z_\lambda)=e(n!)-e((n-A)!)+e(z_{\lambda^*})$, in which $e(z)$ is the power to which a given prime p divides the degree z of the representation concerned of the symmetric group, z_λ is the degree of the irreducible representation associated with the right diagram (partition) λ of n, z_{λ^*} is the degree of the reducible representation corresponding to Robinson's star diagram λ_p^* of λ [Amer. J. Math. **70**, 277–294 (1948); these Rev. **10**, 678], and A denotes the number of nodes of the p-core of λ. The proof involved a formula of Nakayama's depending upon the use of the diagram's p-series, a concept extraneous to the formula (A). The author proves the formula (A) without recourse to the p-series and thereby brings into proper perspective the significance of star diagrams. In § 1 the author provides a useful summary of the numerous definitions employed. In § 2 he proves the existence theorem B: Given a right diagram λ, and a positive integer q, there exists a star diagram λ^*, such that there is a one-to-one correspondence between Nakayama's "hooks" of length kq belonging to λ and those of length k belonging to λ^*. In fact, λ^* represents the actual q-hook structure of λ. If the right diagram λ has k columns containing respectively $\beta_1, \cdots, \beta_k$ nodes then the δ-numbers of λ are given by $\delta_i=\beta_i+k-i$, $i=1, \cdots, k$. In § 3 the structure of λ^* is elucidated by theorem C: Gather the δ-numbers of λ into classes congruent modulo q. For each such class of congruent δ's form the right diagram whose δ's are the δ's of this class. The star diagrams of the diagrams thus formed are the constituents of λ^*. In § 4, A is deduced from C. In § 5 the p-series of λ is deduced from A and a new proof of Nakayama's p-series formula for $e(z)$ is obtained. The proofs involve technicalities which cannot well be summarised here.

D. E. Rutherford (St. Andrews).

Kinosita, Yosihisa. On an enumeration of certain subgroups of a p-group. J. Osaka Inst. Sci. Tech. Part I. **1**, 13–20 (1949).

A formula is obtained expressing the number of subgroups of given type of an Abelian prime-power group in terms of the type invariants. This formula in a slightly different form has also been obtained by Y. Yeh [Bull. Amer. Math. Soc. **54**, 323–327 (1948); these Rev. **9**, 492] and S. Delsarte [Ann. of Math. (2) **49**, 600–609 (1948); these Rev. **10**, 9]. The author derives a polynomial identity involving these numbers of subgroups. From this he deduces an "enumeration principle" for prime-power groups, generalising a result of the reviewer [Zassenhaus, Lehrbuch der Gruppentheorie I, Teubner, Leipzig-Berlin, 1937, p. 116, Satz 18]: if S is any set of elements of the prime-power group G, this is a formula connecting the numbers $N(H)$ of elements of S which lie in the various subgroups H of G which contain the commutator subgroup of G.

P. Hall.

Lyapin, E. On the decomposition of Abelian groups into direct sums of rational groups. Amer. Math. Soc. Translation no. 7, 56 pp. (1950).

Translated from Rec. Math. [Mat. Sbornik] N.S. **8(50)**, 205–237 (1940); these Rev. **3**, 195.

De Groot, J. Exemple d'un groupe avec deux générateurs, contenant un sous-groupe commutatif sans un système fini de générateurs. Nieuw Arch. Wiskunde (2) **23**, 128–130 (1950).

The group is that with generators a, b, and relation $b^2=aba^{-1}$. The subgroup R is that generated by all a^nba^{-n} for integers n. The representation $a:x\rightarrow 2x$, $b:x\rightarrow x+1$ (x real) yields $a^nba^{-n}:x\rightarrow x+2^n$, whence R is isomorphic to the additive group of all dyadic rationals. A similar example, by van der Waerden, takes $a:z\rightarrow \alpha z$, $b:z\rightarrow z+1$ (z complex), where α is transcendental. The corresponding R is free Abelian of infinite rank. [Note: a and b have been erroneously interchanged on p. 130.]

R. C. Lyndon.

MacLane, Saunders. Cohomology theory in abstract groups. III. Operator homomorphisms of kernels. Ann. of Math. (2) **50**, 736–761 (1949).

Ce travail se rattache à 2 mémoires antérieurs de l'auteur en collaboration avec S. Eilenberg [mêmes Ann. (2) **48**, 51–78, 326–341 (1947), numérotés I et II; ces Rev. **8**, 367; **9**, 7]. On y étudie une situation algébrique, qu'on rencontre aussi bien dans l'étude des Q-noyaux [mémoire II] que dans l'étude de l'invariant k^3 d'un polyèdre [cf. Eilenberg et l'auteur, Proc. Nat. Acad. Sci. U. S. A. **32**, 277–280 (1946); Trans. Amer. Math. Soc. **65**, 49–99 (1949); ces Rev. **8**, 398; **11**, 379].

Notations: pour tout groupe K (abélien ou non), $A(K)$ désigne le groupe des automorphismes de K, $I(K)$ le sous-groupe des automorphismes intérieurs; on a un homomorphisme naturel $K\rightarrow I(K)$. Un groupe Q opère dans K si on s'est donné un homomorphisme $Q\rightarrow A(K)$; notons K^Q le groupe K muni de son groupe d'opérateurs Q. Si G est un groupe abélien dans lequel opère aussi Q, notons

Hom (K^Q, G^Q) le groupe (abélien) des homomorphismes de K dans G (isomorphe au groupe des homomorphismes de K_0 dans G, où K_0 est le quotient de K par le sous-groupe des commutateurs), muni du groupe d'opérateurs Q comme suit: un élément $x\varepsilon Q$ transforme un homomorphisme $k\rightarrow f(k)$ dans l'homomorphisme $k\rightarrow x\cdot f(x^{-1}\cdot k)$. Notons Ophom (K^Q, G^Q) le sous-groupe de Hom (K^Q, G^Q) formé des éléments invariants par Q. Enfin, on notera Hom (Q, G^Q) le groupe (abélien) des "homomorphismes croisés" de Q dans G, c'est-à-dire des applications f telles que $f(xy)=f(x)+x\cdot f(y)$.

On étudie la structure (S, η) définie par la donnée: (1) d'une suite exacte de groupes et d'homomorphismes $H\rightarrow K\rightarrow F\rightarrow Q$ (où le premier homomorphisme est biunivoque, et le dernier est sur; on identifie H à un sous-groupe de K); (2) d'un homomorphisme $\eta: F\rightarrow A(K)$ "compatible" avec S, c'est-à-dire vérifiant les 2 conditions suivantes: (i) on a le diagramme de compatibilité

$$\begin{array}{ccccc} H & \rightarrow & K & \longrightarrow & F & \rightarrow & Q \\ & & \downarrow & & \downarrow & & \\ & & I(K) & \longrightarrow & A(K) & & \end{array}$$

(ce qui implique que H est dans le centre de K; on note H additivement); (ii) F opérant dans F par $F\rightarrow I(F)$, l'homomorphisme $K\rightarrow F$ est compatible avec les opérateurs de F (dans K et dans F).

La condition (ii) est automatique si H est le centre de K; il y a correspondance biunivoque entre les systèmes (S, η) vérifiant cette condition, et les groupes K (de centre H) munis d'une structure de Q-noyau [cf. II]. Les résultats qui suivent s'appliquent donc aux Q-noyaux. On a aussi un système (S, η) en considérant, dans un polyèdre connexe P, la suite d'homotopie relative à P et à son squelette P^1 de dimension un (ce qui donne une nouvelle interprétation, suggérée par J. A. Zilber et D. Gorenstein, de l'invariant k^3 d'Eilenberg-MacLane).

Soit R le sous-groupe de F, image de K (Q s'identifie à F/R, et R à K/H). Soit R_0 le groupe R rendu abélien. L'homomorphisme η définit $Q\rightarrow A(H)$; on définit, comme en II, un élément du groupe de cohomologie $H^3(Q, H^Q)$: l'"obstruction" de (S, η). L'auteur introduit aussi la "déviation" de (S, η), dans le cas où H est facteur direct de K (ce qui est notamment le cas lorsque F est un groupe libre): la donnée d'un projecteur de K sur H définit un homomorphisme croisé de Q dans Hom (R_0^Q, H^Q), dont la classe est un élément du groupe de cohomologie $H^1(Q, \text{Hom}\,(R_0^Q, H^Q))$, appelé la "déviation" de (S, η). L'auteur donne aussi une nouvelle définition de l'homomorphisme (déjà connu): $H^1(Q, \text{Hom}\,(R_0^Q, H^Q))\rightarrow H^3(Q, H^Q)$ [cf. I], et constate qu'il transforme la déviation de (S, η) dans son obstruction. Cet homomorphisme est un isomorphisme sur, quand F est libre ("cup product reduction theorem"). On en déduit le théorème: la suite S étant donnée (avec un groupe F libre), et l'homomorphisme $Q\rightarrow A(H)$ étant donné, il existe, pour tout élément $\lambda_3\varepsilon H^3(Q, H^Q)$, un homomorphisme η compatible avec S et avec $Q\rightarrow A(H)$, et dont λ_3 soit l'obstruction; en outre, η est unique, à un "isomorphisme" près de la structure (S, η). Ceci entraîne notamment le théorème (connu) d'existence d'un Q-noyau ayant une obstruction donnée.

Soit maintenant G un groupe abélien dans lequel opère Q, donc aussi F. On introduit les groupes Map $(R, F; G^Q)$: quotient de Ophom (R^F, G^F) par l'image de Hom (F, G^F); Map) $S, \eta; G^Q)$: quotient de Ophom (K^F, G^F) par l'image de Hom (F, G^F). On définit les homomorphismes suivants:

$$\begin{array}{l} 0\rightarrow \text{Map}\,(R, F; G^Q)\rightarrow \text{Map}\,(S, \eta; G^Q)\longrightarrow \text{Ophom}\,(H^Q, G^Q) \\ \qquad\qquad\downarrow \qquad\qquad\qquad\qquad \downarrow \qquad\qquad\qquad\qquad \updownarrow \\ 0\longrightarrow H^2(Q, G^Q)\longrightarrow E^2(Q, H^Q, l_3, G^Q)\rightarrow \text{Ophom}\,(H^Q, G^Q) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \rightarrow H^3(Q, G^Q), \end{array}$$

où les suites horizontales sont exactes; $E^2(Q, H^Q, l_3, G^Q)$ désigne un groupe dont la construction explicite est donnée à l'aide d'un cocycle l_3 choisi dans la classe de l'obstruction de (S, η). Tous les homomorphismes du diagramme, sauf ceux se rapportant à E^2, sont indépendants du choix de l_3.

Si en outre H est facteur direct de K, la déviation de (S, η) est définie et permet de compléter le diagramme:

$$\begin{array}{l} 0\rightarrow \text{Map}\,(R, F; G^Q)\rightarrow \text{Map}\,(S, \eta; G^Q)\rightarrow \\ \qquad\qquad\downarrow \qquad\qquad\qquad\qquad \downarrow \\ 0\longrightarrow H^2(Q, G^Q)\longrightarrow E^2(Q, H^Q, l_3, G^Q)\rightarrow \\ \qquad\qquad\qquad\qquad \text{Ophom}\,(H^Q, G^Q)\rightarrow H^1(Q, \text{Hom}\,(R_0^Q, G^Q)) \\ \qquad\qquad\qquad\qquad\qquad \updownarrow \qquad\qquad\qquad\qquad \downarrow \\ \qquad\qquad\qquad\qquad \text{Ophom}\,(H^Q, G^Q)\rightarrow H^3(Q, G^Q). \end{array}$$

Enfin, si F est libre, le premier et le dernier homomorphismes verticaux sont des isomorphismes sur; il en résulte que le second l'est aussi: Map $(S, \eta; G^Q)$ s'identifie à $E^2(Q, H^Q, l_3, G^Q)$, qui dépend seulement (à un isomorphisme près) de la connaissance de Q, H^Q, G^Q, et de l'obstruction de (S, η).

Dans le cas de la structure (S, η) attachée à un polyèdre P et à son squelette P^1, on retrouve, dans E^2, la construction du groupe de cohomologie $H^2(P, G)$ à l'aide de $\pi_1(P)$, $\pi_2(P)$ et de l'invariant k^3. Les résultats précédents ont une traduction duale en "homologie." [Erratum: dans les 4 dernières lignes de la p. 740, remplacer 4 fois la lettre G par H.] *H. Cartan* (Paris).

Wang, Hsien-Chung. A problem of P. A. Smith. Proc. Amer. Math. Soc. **1**, 18–19 (1950).

In connection with a question raised by P. A. Smith [Bull. Amer. Math. Soc. **48**, 309–312 (1942); these Rev. **4**, 3] the author proves the existence of noncountable and dense proper subgroups in every separable, locally-compact, nondiscrete metric group.

H. Freudenthal (Utrecht).

Sherman, Seymour. A problem of Raĭkov. Duke Math. J. **17**, 21–26 (1950).

Invariant measures on locally totally bounded (but not necessarily locally compact, or separable, or commutative) topological groups G are discussed. It is shown that the Haar-Raĭkov measure, i.e., a left invariant measure defined on the σ-ring generated by the family of all countably open sets of G [D. Raĭkov, Trav. Inst. Math. Stekloff **14** (1945); these Rev. **8**, 133] can be extended to a left invariant measure on the σ-ring of all Borel sets of G. The extension is unique for σ-bounded sets. Further, it is shown that, similarly to the case of Weil measure [A. Weil, L'intégration dans les groupes topologiques et ses applications, Actual. Sci. Ind., no. 869, Hermann, Paris, 1940; these Rev. **3**, 198] G is thick in its locally compact completion $\bar{G}$, i.e., $\bar{G}-G$ has inner measure zero with respect to the Haar measure of $\bar{G}$ [cf. K. Kodaira, Proc. Phys.-Math. Soc. Japan (3) **23**, 67–119 (1941); these Rev. **2**, 317]. *S. Kakutani.*

Choudhury, A. C. Quasi-groups and nonassociative systems. II. Bull. Calcutta Math. Soc. **41**, 211–219 (1949).

[For part I cf. the same Bull. **40**, 183–194 (1948); these Rev. **10**, 591.] This paper is concerned with the two relations between two elements a, b of a quasi-group ϕ, defined by the statements "a is a right identity for b," "a is a left identity for b" (i.e., $ba=b$, $ab=b$). These relations can be used to define "graphs" whose nodes are the elements of the quasi-group. The author defines RI-sets and LI-sets to be subsets of ϕ with the property that if a belongs to an RI-set (LI-set) so does the right (left) identity of a. Both classes of subsets form a lattice and the author defines topologies on these lattices.

D. Rees (Manchester).

NUMBER THEORY

Zeckendorf, É. Étude fibonaccienne. De certaines coupes obliques parallèles dans les polytopes arithmétiques à $(p-1)$ dimensions. Mathesis **58**, 293–306 (1950).

The points in Euclidean p-space whose Cartesian coordinates are positive integers with a fixed sum n are symmetrically arranged inside a regular simplex of edge $n\sqrt{2}$. The author considers the distribution of these points in a series of parallel sections, which could be represented by assigning various values to a given linear form in the coordinates.

H. S. M. Coxeter (Toronto, Ont.).

Trypanis, A. A. An extension of Fermat's theorem. Amer. Math. Monthly **57**, 87–89 (1950).

Corresponding to the generalization $a^{p^{n-1}(p-1)}-1\equiv 0 \pmod{p^n}$ of Fermat's theorem, it is proved that, in the integral domain of algebraic integers, $a^{(p-1)/p^n}-1$ is divisible by p^{1/p^n} for any odd prime p and rational integer a not divisible by p. [Reviewer's remark: the result appears to be true for $p=2$ also.]

I. Niven (Eugene, Ore.).

Duarte, F. J. On some theorems of arithmetic. Estados Unidos de Venezuela. Bol. Acad. Ci. Fís. Mat. Nat. **11**, 481–485 (1948). (Spanish)

The author gives simple but known proofs of Fermat's theorem, Wilson's theorem and some theorems related to these.

H. W. Brinkmann (Swarthmore, Pa.).

Salem, R., and Spencer, D. C. On sets which do not contain a given number of terms in arithmetical progression. Nieuw Arch. Wiskunde (2) **23**, 133–143 (1950).

Denote by $\nu_k(x)$ the maximum number of integers not exceeding x which do not contain an arithmetic progression of k terms. It has been conjectured that $\lim_{x\to\infty}\nu_k(x)/x=0$. The proof of this seemingly simple conjecture seems very difficult. It is easy to see on the other hand that $\lim_{x\to\infty}\nu_k(x)/x$ exists. The authors prove that $\nu_k(x)>x^{1-\epsilon}$ for every ϵ if x is sufficiently large. Various analogous problems are discussed.

P. Erdös.

Klobe, W. Über eine untere Abschätzung der n-ten Kreisteilungspolynome

$$g_n(z)=\prod_{d/n}(z^d-1)^{\mu\left(\frac{n}{d}\right)}.$$

J. Reine Angew. Math. **187**, 68–69 (1949).

The fact that the polynomial $g_n(z)$ defined in the title is less than $z-1$ for z real and greater than 1 was used by E. Witt in his proof of Wedderburn's theorem [Abh. Math. Sem. Hamburg. Univ. **8**, 413 (1931)]. The present paper gives a "purely arithmetical" proof of this inequality, no use being made of the complex linear factors of the cyclotomic polynomial.

D. H. Lehmer (Berkeley, Calif.).

Ryde, Folke. Fast-monotone Kettenbrüche. Ark. Mat. **1**, 27–44 (1949).

A continued fraction of the form

$$a_1+\frac{a_1}{a_2+}\,\frac{a_2}{a_3+}\cdots+\frac{a_{n-1}}{a_n}\,\frac{a_n}{+a_{n+1}},$$

where the positive integers a_i $(i=1,\cdots,n+1)$ are such that $a_1\geqq a_2\geqq\cdots\geqq a_n$ is called monotonic nonincreasing if $a_{n+1}=1$ and almost monotonic if $2a_n<a_{n+1}$. In previous papers [Ark. Mat. Astr. Fys. **31A**, no. 19 (1945); **31B**, no. 12 (1945); **34A**, nos. 11, 16 (1947); these Rev. **8**, 5; **9**, 269] the author has obtained various results for monotonic nonincreasing continued fractions. In the present papers, corresponding results are obtained for almost monotonic continued fractions. It is shown, for example, that no rational number between 1 and 3 may be expressed in more than one way by such a continued fraction. It is also shown that necessary and sufficient conditions that a rational r may be developed in the form of an almost monotonic continued fraction may be expressed in terms of properties of certain factors of r. The author also obtains recursion formulas for $Nm'(r)$, the number of such developments, and concludes with a table for $Nm'(r)$, where $r=(p-q)/q$ and p, q are positive integers with $p\leqq 50$ and $q\leqq p-1$.

W. H. Gage (Vancouver, B. C.).

David, Marcel. Sur un algorithme voisin de celui de Jacobi. C. R. Acad. Sci. Paris **229**, 965–967 (1949).

The author states that if α_0 is any irrational element of a real cubic extension of the rational field, all of whose conjugate fields are also real, then there exists a number β_0 in the same field such that the sequences of rational integers $\bar{a}_n, b_n$ defined by the equations

$$\alpha_n=\bar{a}_n-\beta_{n+1}/\alpha_{n+1};\quad \beta_n=b_n+1/\alpha_{n+1};\qquad n=0, 1, 2, \cdots$$

(where $\bar{a}_n$ is the least integer greater than α_n, and $b_n=[\beta_n]$), are not periodic, and outlines a geometric proof. This is of interest because of its similarity to Jacobi's [J. Reine Angew. Math. **69**, 29–64 (1868)] development $\alpha_n=a_n+\beta_{n+1}/\alpha_{n+1}$; $\beta_n=b_n+1/\alpha_{n+1}$, $a_n=[\alpha_n]$, for which it is still not known whether the sequence of a_n, b_n which it defines is or is not periodic for all α_0, β_0 in a cubic field of the above type.

G. Whaples (Bloomington, Ind.).

Rédei, L. Über die Anzahl der Potenzreste mod p im Intervall 1, $\sqrt{p}$. Nieuw Arch. Wiskunde (2) **23**, 150–162 (1950).

Let p be a prime greater than 3 and l an integer greater than unity such that $l|(p-1)$. Let the integers prime to p be divided into l classes $C_0, \cdots, C_{l-1}$ in such a way that two integers are in the same class if and only if their quotient modulo p is an lth-power residue modulo p. Under the additional restriction that l divides $\frac{1}{2}(p-1)$ the author proves that the number of integers in the class C_i $(i=0, 1, \cdots, l-1)$ which lie in the interval 1, $\sqrt{p}$ is less

than $\{1-(2+\sqrt{2})^{-1}(1-1/l)\}[\sqrt{p}]$. For l a divisor of $p-1$ but not of $\frac{1}{2}(p-1)$ the author considers only the case $l=2$; he proves that for $p\equiv 3 \pmod 4$ the number of quadratic residues and the number of quadratic nonresidues which lie in the interval 1, $2\sqrt{(p/3)}$ are both less than $\{1-(8+4\sqrt{3})^{-1}\}[2\sqrt{(p/3)}]$. The proofs are based on the following result of the author which is in the course of publication: If K is a convex domain in the Cartesian (x, y)-plane which is symmetrical in the origin, has area $4p$, and is contained in the open square $|x|, |y|<p$, then modulo p the ratio y/x represents all nonzero residue classes modulo p when x and y run over all pairs of nonzero integers such that (x, y) is in K. *P. T. Bateman.*

Wachs, Sylvain. Contribution à l'étude de l'irrationalité de certains nombres. Bull. Sci. Math. (2) **73**, 77–95 (1949).

In part I of this contribution the author gives a condition for the irrationality of the sum of an infinite series of rational terms. [This theorem is well-known; see Koksma, Diophantische Approximationen, Springer, Berlin, 1936, p. 54.] In part II he starts from the integral

$$(1)\qquad \int_0^\pi x^n(\pi-x)^n \sin x dx,$$

introduced by Niven to prove the irrationality of π [Bull. Amer. Math. Soc. **53**, 509 (1947); these Rev. **9**, 10]. By considering a larger class of similar integrals the author intends to obtain more general results. He gives applications by showing in this manner the irrationality of such numbers as π^2, $\log A$ $(A\neq 1)$, e^A, where A denotes a positive integer. [The paper contains some misprints and other mistakes; the most serious one at the end of § 6, where the quantity M introduced depends on n. In view of various papers giving generalizations of Niven's method it is perhaps of interest to remark that there exists a close connection between this method and the classical proofs for the irrationality of π and π^2 of Lambert, Hermite and others. Take for instance the integral used by Jordan [Cours d'Analyse, v. 2, 3rd ed., Gauthier-Villars, Paris, 1913, p. 106] for this purpose, i.e.,

$$\int_{-1}^{1}(1-t^2)^n \cos ztdt.$$

Putting $z=\pi/2$, $1+t=2x/\pi$, we obtain, apart from the factor $(2/\pi)^{2n+1}$, the integral (1).] *J. Popken.*

✱Verdenius, W. On generalized Gauss sums. Handelingen van het XXXI^e Nederlands Natuur- en Geneeskundig Congres, pp. 93–94, Haarlem, 1949. (Dutch)

The following theorem is stated without proof. Let $\psi(y_1, \cdots, y_s)$ be a quadratic form (not necessarily homogeneous) with integral coefficients and assume the discriminant of the quadratic part to be different from 0. Then there exists a constant $C>0$ such that for any prime p, any $\alpha\geqq 1$ and any t the number $N(p^\alpha, t)$ of the solutions of $\psi(y_1, \cdots, y_s)\equiv t \pmod{p^\alpha}$ is either zero or greater than $Cp^{(s-1)\alpha}$. The same holds if the condition $p\nmid y_1\cdots y_s$ is added. The author states that he obtained these results by evaluation of generalised Gauss sums. *N. G. de Bruijn.*

Schneider, Theodor. Eine Bemerkung zur Minkowskischen Vermutung über inhomogene Linearformen. Arch. Math. **2**, 87–89 (1950).

Let $\xi_1, \cdots, \xi_n$ be n homogeneous linear forms in n variables x_i, with determinant Δ. Let $\eta_1, \cdots, \eta_n$ be n real numbers. Then a conjecture of Minkowski asserts that integral x_i exist such that (1) $\prod_1^n|\xi_i-\eta_i|\leqq 2^{-n}|\Delta|$. The author proves (1) under the following additional hypothesis: there exist positive real t_i with $\prod_1^n t_i=|\Delta|$, such that no integral values of the x_i not all zero satisfy the inequalities $(|\xi_i|/t_i)<1$, $i=1, 2, \cdots, n$, and (2) $\sum_1^n(|\xi_i|/t_i)<\frac{1}{2}n$. The proof is short and elementary. A previous theorem of S. Kowner [Rec. Math. [Mat. Sbornik] (1) **32**, 528–541 (1925)] proved (1) on the stronger hypothesis obtained from the above by omitting condition (2).

F. J. Dyson (Birmingham).

Mullender, P. Lattice points in non-convex regions. III. Nederl. Akad. Wetensch., Proc. **52**, 18–28 = Indagationes Math. **11**, 50–60 (1949).

Continuing two previous notes the author gives the proof of the theorem contained in part II B [same Proc. **51**, 1251–1261 = Indagationes Math. **10**, 395–405 (1948); these Rev. **10**, 593] and applies them to the following cases [see the notation in the cited review]:

$$(1)\qquad F(x_1, \cdots, x_n)=(X_1^{p_1}\cdots X_r^{p_r})^{1/(p_1+\cdots+p_r)},$$

where $X_i=p_i^{-1}\sum_{k=1}^{p_i}|x_{p_1+\cdots+p_{i-1}+k}|$, with $p_1+\cdots+p_r=n$;

$$(2)\qquad F(x_1, \cdots, x_n)=(|x_1|^\sigma+\cdots+|x_n|^\sigma)^{1/\sigma},$$

where $0<\sigma<1$. *V. Knichal* (Prague).

Chabauty, Claude. Sur les minima arithmétiques des formes. Ann. Sci. École Norm. Sup. (3) **66**, 367–394 (1949).

Let $f(x)$ be a function, defined at all points $x=(x_1, \cdots, x_n)$ of the Euclidean space R^n, and satisfying $f(tx)=|tf(x)|$ for all real t and for all points x. Let G be a lattice in R^n with determinant $m(G)$. For $h=1, \cdots, n$, the hth arithmetical minimum $\mu_h=\mu_h(f, G)$ of f for G is defined to be the lower bound of the numbers μ such that there are at least h linearly independent points x of G with $f(x)\leqq\mu$. Write $\gamma(f)=\sup \mu_1(f, G)\{m(G)\}^{-1/n}$, the upper bound being taken over all lattices G. [Then in Mahler's notation

$$\Delta(S)=\{\gamma(f)\}^{-n},$$

where S is the set defined by $f(x)\leqq 1$.] The author gives detailed proofs of his main results that (a) for every such function f and every lattice G,

$$(*)\qquad \mu_1\cdots\mu_n\leqq 2^{\frac{1}{2}(n-1)}m(G)\{\gamma(f)\}^n,$$

and (b) that this inequality is satisfied with equality in certain special cases. He had previously given concise proofs of these results [C. R. Acad. Sci. Paris **228**, 796–797 (1949); these Rev. **10**, 511]. See also papers by the reviewer [Nederl. Akad. Wetensch., Proc. **52**, 256–263 = Indagationes Math. **11**, 71–78 (1949); these Rev. **10**, 511] and by K. Mahler [ibid., 633–642 = Indagationes Math. **11**, 195–204 (1949); these Rev. **10**, 512], giving independent proofs of (a) and of (b), respectively.

The convexity coefficient $\omega(f)$ of the function f is defined to be the upper bound of $f(x+y)/\{f(x)+f(y)\}$ taken over all pairs of points, not both coinciding with the origin. Let $V(f)$ be the interior Lebesgue measure of the set S given by $f(x)\leqq 1$. Let φ be a positive definite homogeneous polynomial in $x_1, \cdots, x_n$ of degree d with integral coefficients. By application of (*) to the function $f=|\varphi|^{1/d}$ it is shown that the number of arithmetically inequivalent polynomials φ of this form, having $n+d+\omega(f)+\{V(f)\}^{-1}$ less than any given constant, is finite.

The result (*) is used to generalize results of Khintchine [Math. Ann. **113**, 398–415 (1936)] and Mahler [Nederl.

Akad. Wetensch., Proc. 41, 634–637 (1938)] by establishing a connection between the homogeneous and nonhomogeneous problems for bounded star bodies. Among the other topics discussed are results corresponding to (*) for star bodies and for arbitrary sets. *C. A. Rogers.*

***van der Corput, J. G. On the regularity of large numbers.** Handelingen van het XXXI° Nederlands Natuur- en Geneeskundig Congres, pp. 36–52, Haarlem, 1949. (Dutch)

Lecture on the distribution of primes and computation of series.

Rényi, Alfred. On a theorem of Erdös and Turán. Proc. Amer. Math. Soc. 1, 7–10 (1950).

The theorem referred to in the title states that, if p_n denotes the nth prime, then the sequence $\log p_n$ is neither convex nor concave for all sufficiently large n, that is, the sequence $p_{n+1}p_{n-1}-p_n^2$ changes sign infinitely often [Erdös and Turan, Bull. Amer. Math. Soc. 54, 371–378 (1948); these Rev. 9, 498]. The present paper establishes the following more general result. The total curvature $G_N=\sum_{k=2}^{n-1}|\arg(z_{k+1}-z_k)/(z_k-z_{k-1})|$ of the polygonal line having vertices $z_k=k+i\log p_k$, $p_n\leqq N$, exceeds $c\log\log\log N$, where c is a positive constant. It is clear that if the sequence $\log p_n$ were convex (or concave) for all sufficiently large n, the total curvature would be finite. The proof uses the prime number theorem in the form $\pi(n)\sim n/\log n$, and the result $p_{n+1}-p_n=O(p_n/\log^2 p_n)$. It is shown first of all that $G_N\geqq c_3\sum_{p_{n+1}\leqq N}(\Delta_n/p_n)-c_4$, where $\Delta_n=p_{n+1}-2p_n+p_{n-1}$ and c_3, c_4 are positive constants. A lower bound for the sum is then found by estimating the number of primes $p_{n+1}\leqq N$ for which $\Delta_n\neq 0$. This leads to the final result. The author remarks that whether there exist blocks of primes $p_{r_\nu}, p_{r_\nu+1}, \cdots, p_{r_\nu+k_\nu}$ with $\Delta_{r_\nu+\nu}=0$, $\nu=1,2,\cdots,k_r$, of any length k_r and whether there exist infinitely many k for which $\Delta_k=0$ are unsolved problems. *R. D. James.*

Selberg, Atle. An elementary proof of the prime-number theorem for arithmetic progressions. Canadian J. Math. 2, 66–78 (1950).

This is an elementary proof of the formula $\vartheta_{k,l}(x)\sim x/\varphi(k)$, where $(k,l)=1$, and $\vartheta_{k,l}(x)=\vartheta_l(x)=\sum\log p$ summed over the range $p\leqq x$, $p\equiv l \pmod k$ (p prime). The proof follows the general lines of the author's treatment of the case $k=1$ [Ann. of Math. (2) 50, 305–313 (1949); these Rev. 10, 595; quoted as (ii)]. The starting point is the formula

$$(1)\quad \sum_{p\leqq x;\, p\equiv l\,(k)}\log^2 p+\sum_{pq\leqq x;\, pq\equiv l\,(k)}\log p\log q=\frac{2}{\varphi(k)}x\log x+O(x)$$

(p, q prime). From this it is first deduced that

$$(2)\quad |R_l(x)|\leqq\frac{1}{\varphi(k)\log x}\sum_\alpha\sum_{n\leqq x}|R_\alpha(x/n)|+O\left(\frac{x\log\log x}{\log x}\right),$$

$$(3)\quad \sum_{n\leqq x}n^{-1}\log n R_l(n)=-\sum_{n\leqq x}n^{-1}\sum_{\alpha\beta\equiv l\,(k)}R_\alpha(n)R_\beta(x/n)+O(x),$$

where $R_l(x)=\vartheta_l(x)-x/\varphi(k)$, and α, β run over reduced sets of residues (mod k). The general idea is then to show that $|R_l(x)/x|<\sigma_n/\varphi(k)$ $(x>x_n)$, where σ_n satisfies a recurrence relation implying that $\sigma_n\to 0$ as $n\to\infty$. But the extension to $k>1$ brings new difficulties. One of these entails the introduction of (3) in place of a more elementary result that sufficed when $k=1$. But the main difficulty arises from the need for a sufficiently elementary proof that $\vartheta_l(x)/x>c(k)>0$ $(x>x_0)$. This the author resolves, with the help of a little of the theory of real (but not complex) characters, by borrowing and developing ideas from his elementary treatment of Dirichlet's theorem on the existence of primes in arithmetical progressions [Ann. of Math. (2) 50, 297–304 (1949); these Rev. 10, 595; quoted as (i)].

The proof of (1) itself is pieced together from the author's papers (i) and (ii). In (i) it was proved that $T_{k,l}(x)=S_k(x)+O(x)$, where $T_{k,l}(x)$ is the left hand side of (1) and $S_k(x)$ is independent of l; from which it was deduced, by summation over l, that $\varphi(k)T_{k,l}(x)=T_{1,0}(x)+O(x)$. The problem of the asymptotic behaviour of $T_{k,l}(x)$ was thus reduced to the case $k=1$; and this was dealt with in (ii) by an elementary estimation of $S_1(x)$. The actual form of $S_k(x)$ used was

$$S_k(x)=xk^{-1}\sum_{d\leqq x;\,(d,k)=1}d^{-1}\mu(d)\log^2(x/d).$$

(This is equivalent, to order $O(x)$, to the sum of those coefficients b_n in $\zeta''(s)/\zeta(s)=\sum b_n n^{-s}$ for which $n\leqq x$, $n\equiv l$ (mod k), but was discussed by the author without reference to Dirichlet's series.) *A. E. Ingham.*

Shapiro, Harold N. Some assertions equivalent to the prime number theorem for arithmetic progressions. Comm. Pure Appl. Math. 2, 293–308 (1949).

It is shewn that the prime number theorem for primes $p\equiv B \pmod A$, where $(A,B)=1$, may be stated in the following equivalent forms:

$$(1)\quad \pi(A,B,x)\sim x/\{\varphi(A)\log x\},$$

$$(2)\quad \theta(A,B,x)\sim x/\varphi(A),$$

$$(3)\quad \psi(A,B,x)\sim x/\varphi(A),$$

$$(4)\quad \sum_{n\equiv B\,(\mathrm{mod}\,A)}\mu(n)/n \text{ converges},$$

$$(5)\quad M_B(x)=\sum_{n\leqq x;\,n\equiv B\,(\mathrm{mod}\,A)}\mu(n)=o(x),$$

$$(6)\quad H_B(x)=\sum_{n\leqq x;\,n\equiv B\,(\mathrm{mod}\,A)}\mu(n)n^{-1}\log n=o(\log x).$$

These results are the analogues of similar results for the ordinary prime number theorem derived by Landau [see, for example, Handbuch der Lehre von der Verteilung der Primzahlen, v. 2, Teubner, Leipzig-Berlin, 1909], the word "equivalent" bearing the same interpretation as that given by Landau; i.e., no use is made of complex variable theory or any other "transcendental" methods. That (1), (2) and (3) are equivalent is well known; the author proves that (3)→(6)→(4)→(5)→(3). The fact that $L(1,\chi)\neq 0$ is used. *R. A. Rankin* (Cambridge, England).

Shapiro, Harold N. On a theorem of Selberg and generalizations. Ann. of Math. (2) 51, 485–497 (1950).

For integral $k\geqq 2$ write

$$V_k(x)=\sum_{n\leqq x}d^{-1}\mu(d)\log^k(x/d).$$

By successive applications of the Möbius inversion formula it is shown that $V_2(x)=2\log x+O(1)$, and from this result Selberg's lemma [Ann. of Math. (2) 50, 305–313 (1949); these Rev. 10, 595]

$$\sum_{p\leqq x}\log^2 p+\sum_{pq\leqq x}\log p\log q=2x\log x+O(x)$$

is deduced (p and q denote primes). Both results are then generalised (induction is used and the inversion formula is

applied to $f(x)=x\log^{k-1}x$) to prove that

$$V_k(x)=k\log^{k-1}x+\sum_{i=1}^{k-2}c_i^{(k)}\log^i x+O(1),$$

where the $c_i^{(k)}$ are constants depending on k, and that

$$\frac{(k+1)!\theta_{k+2}(x)}{x\log^{k+1}x}+\frac{k!\theta_{k+1}(x)}{x\log^k x}=2+O(1/\log x),$$

where

$$\theta_r(x)=\sum\log p_1\log p_2\cdots\log p_r$$

summed over $p_1p_2\cdots p_r\leq x$. By means of this result $\theta_k(x)$ can be expressed in terms of $\theta_1(x)=\theta(x)$ and a further generalisation to sums

$$\sum_{p_1p_2\cdots p_k\leq x}\log^{m_1}p_1\log^{m_2}p_2\cdots\log^{m_k}p_k$$

is proved. If an asymptotic equality for any of these functions can be obtained it will imply $\theta(x)\sim x$, i.e., the prime number theorem. *R. A. Rankin* (Cambridge, England).

Erdös, P. On a Tauberian theorem connected with the new proof of the prime number theorem. J. Indian Math. Soc. (N.S.) 13, 131–144 (1949).

This is a further contribution to the logic of the prime number theorem [see A. Selberg, Ann. of Math. (2) **50**, 305–313 (1949); P. Erdös, Proc. Nat. Acad. Sci. U. S. A. **35**, 374–384 (1949); these Rev. **10**, 595]. It is proved [theorem 1] that, if $1<p_1<p_2<\cdots$ and

$$(1)\qquad \sum_{p_i\leq x}(\log p_i)^2+\sum_{p_ip_j\leq x}\log p_i\log p_j=2x\log x+O(x),$$

then $\vartheta(x)=\sum_{p_i\leq x}\log p_i\sim x$. The effect of this, when the p_i are the primes, is that from Selberg's asymptotic formula in its sharper form we can deduce the prime number theorem without assuming independently any results from the elementary theory of primes. It is deduced, in fact, from (1) that (2) $\sum_{p_i\leq x}p_i^{-1}\log p_i=\log x+O(1)$; and theorem 1 then follows from the earlier work of Selberg and Erdös. The deduction of (2) from (1) is based on the Tauberian theorem [theorem 2]: let $a_k\geq 0$, $s_m=\sum_{k=1}^m a_k$, and suppose $S(n)=\sum_{k=1}^n a_k(s_{n-k}+k)=n^2+O(n)$; then $s_n=n+O(1)$. The proof of this theorem is elementary, but too intricate for a brief summary. The main difficulty arises from the sharpness of the final error term; to illustrate this the author begins by discussing in succession the simpler versions with final errors $o(n)$, $o(n^\epsilon)$, $O(\log n)$. A generalisation [theorem 2′] is stated, in which the errors in hypothesis and conclusion are replaced by $O[nf(n)]$ and $O[f(n)]$, where $f(n)>c>0$, $f(n)/n\to 0$. *A. E. Ingham* (Cambridge, England).

Erdös, P. Supplementary note. J. Indian Math. Soc. (N.S.) 13, 145–147 (1949).

This is a discussion of some further points arising out of theorem 2 of the paper reviewed above. Use the notation of that theorem, and write

$$S(n)=\sum_{k=1}^n a_ks_{n-k}+\sum_{k=1}^n ka_k=S_2(n)+S_1(n),$$

say. The author proves: (i) if $a_k\geq 0$ and $S_1(n)=\frac{1}{2}n^2+O(n)$, then $s_n=n+O(\log n)$; (ii) if $a_k\geq 0$ and $S_2(n)=\frac{1}{2}n^2+O(n)$, then $s_n=n+o(n)$. It is shown by examples that in (i) the final error cannot be improved, while in (ii) it certainly cannot be improved to $o(n^{\frac{1}{2}})$. (This emphasises the distinctive feature of theorem 2; if we separate the two constituents of $S(n)$, we no longer obtain anything sufficiently precise for the prime number theorem.) Another example shows that the conclusion of theorem 2 cannot be improved to $s_n=n+o(1)$ even if we replace $O(n)$ by $O(1)$ in the hypothesis. A generalisation of theorem 2 is given, but one that is quickly reduced to the special case. [Note: (i) is true, by partial summation, without the condition $a_k\geq 0$.] *A. E. Ingham* (Cambridge, England).

Bellman, Richard. An analog of an identity due to Wilton. Duke Math. J. 16, 539–545 (1949).

The author proves the following theorem. For $\sigma>\frac{1}{4}$, $\sigma'>\frac{1}{4}$, $\sigma+\sigma'>1$,

$$\begin{aligned}\zeta^2(s)\zeta^2(s')-\zeta'(s+s'-1)\{(s-1)^{-1}+(s'-1)^{-1}\}\\ +\zeta^2(s+s'-1)\{(s-1)^{-2}+(s'-1)^{-2}+2C(s-1)^{-1}\\ +2C(s'-1)^{-1}\}=2\sum_{r=1}^{\infty}\sum_{nk=r}d(k)d(n)n^{-s-s'+1}\\ \times\int_1^\infty M_0(4\pi r^{\frac{1}{2}}u)(u^{-2s+1}+u^{-2s'+1})du,\end{aligned}$$

where $\zeta(s)$ is the Riemann ζ-function, C Euler's constant, $d(n)$ the number of divisors of n, and $M_0(u)=-Y_0(u)+2\pi^{-1}K_0(u)$ in the notation normally used for Bessel functions. This theorem is a generalization of a result by Wilton [Proc. London Math. Soc. (2) 31, 11–17 (1930)], who proved a similar identity for $\zeta(s)\zeta'(s)$. The proof, based on Voronoi's summation formula, is complicated by considerations of convergence. The result is best possible in so far as the infinite series diverges for $\sigma=\sigma'=\frac{1}{4}$. *H. A. Heilbronn.*

Flett, T. M. On the function

$$\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{t}{n}.$$

J. London Math. Soc. 25, 5–19 (1950).

Let $Q(t)=\sum_{n=1}^{\infty}n^{-1}\sin(t/n)$, $P(t)=\sum_{n<t}n^{-1}\cos(t/n)$. Hardy and Littlewood proved that both $P(t)$ and $Q(t)$ are unbounded. The author proves that both $Q(t)$ and $P(t)$ are $O[(\log t)^{\frac{3}{4}}(\log\log t)^{\frac{1}{4}+\epsilon}]$. In the proof the author uses results of van der Corput and Tchudakoff. The author also improves the O results for $\zeta(1+it)$. *P. Erdös.*

Chowla, S. A new proof of a theorem of Siegel. Ann. of Math. (2) 51, 120–122 (1950).

The author gives a slightly simplified proof of Siegel's theorem [Acta Arith. 1, 83–86 (1935)] that $L(1,\chi)>C(\epsilon)k^{-\epsilon}$, where χ is a real character mod k. See also Estermann [J. London Math. Soc. 23, 275–279 (1948); these Rev. 10, 356]. *H. A. Heilbronn* (Bristol).

Stoll, W. Eine Bemerkung über die Dirichletschen *L*-Reihen. Math. Z. 52, 307–309 (1949).

The author proves a relation which is essentially the identity ($p\equiv 3\pmod 4$ a prime)

$$\pi^{-1}p^{\frac{1}{2}}\sum_{n=1}^{\infty}(n/p)n^{-1}=-\sum_{n=1}^{p-1}n(n/p).$$

Since the finite sum is odd, the infinite series is different from zero. All this was known to Dirichlet [J. Reine Angew. Math. 21, 134–155 (1840)]. *H. A. Heilbronn* (Bristol).

Petersson, Hans. Über den Körper der Fourierkoeffizienten der von Hecke untersuchten Eisensteinreihen. Abh. Math. Sem. Univ. Hamburg 16, 101–113 (1949).

The series in question are

$$G_r(\tau, a_1, a_2, N) = \sum_{m_i \equiv a_i(N)}' (m_1\tau+m_2)^{-r},$$

$$G_r^*(\tau, a_1, a_2, N) = \sum_{m_i \equiv a_i(N)}^* (m_1\tau+m_2)^{-r},$$

where $\sum'$ denotes that the term $m_1=m_2=0$ is to be omitted and $\sum^*$ that only the terms with $(m_1, m_2)=1$ are to be taken (N and r are positive integers). If $\tau'=A\tau$ is a modular substitution which transforms a vertex ζ of the fundamental region of $\Gamma(N)$ (the principal congruence group modulo N) to ∞, then the functions G_r and G_r^* can be expanded in Fourier series in the uniformising variable $\exp 2\pi i A\tau/N$ of ζ. The author proves that for the G_r^* all Fourier coefficients lie, for all vertices ζ, in the field of the Nth roots of unity. If N and r are fixed and analytic entire modular forms, not equal to 0, with the given N and r exist, then all coefficients have a common ideal denominator. The same is true for the functions $(-2\pi i)^{-r}N^r(r-1)!G_r(\tau, a_1, a_2, N)$ if $(a_1, a_2, N)=1$. If $F(\tau)$ is a linear combination with constant coefficients of the G_r^* (N and r fixed) then all Fourier coefficients of $F(\tau)$ in all vertices ζ lie in the field obtained by adjunction of $\exp 2\pi i/N$, and of the constant terms in all Fourier expansions of $F(\tau)$, to the field of rational numbers (if $r\geqq 2$; slight modification for $r=1$).

H. D. Kloosterman (Leiden).

vol 3-4

Petersson, Hans. Über die Werte der Riemannschen Zetafunktion für positive ungerade Argumente. Abh. Math. Sem. Univ. Hamburg 16, 119–135 (1949).

Denote by $C(q)$ ($q\geqq 2$, a prime) the linear set of all cusp forms (entire modular forms vanishing at all parabolic vertices of the fundamental region) of dimension $-r$ belonging to the group consisting of all modular substitutions $\tau\rightarrow(\alpha\tau+\beta)(\gamma\tau+\delta)^{-1}$ with $\gamma\equiv 0 \pmod q$. The author proves that two entire modular forms of dimension $-r$ ($r\geqq 4$, even) which [in the sense of the metric introduced by the author in Math. Ann. 117, 453–537 (1940); these Rev. 2, 87] are orthogonal to all forms of $C(q)$ and which possess a scalar product ω (i.e., if their product is a cusp form) must, except for numerical factors, be identical with the two functions $E_1=\sum_1(m_1\tau+m_2)^{-r}$, $E_2=\sum_2(m_1\tau+m_2)^{-r}$ where $\sum_1$ and $\sum_2$ denote summations over all pairs of rational integers satisfying (1) $m_1\equiv 0 \pmod q$, $(m_2, q)=1$ and (2) $(m_1, q)=1$, respectively. The scalar product $\omega=(E_1, E_2)$ of E_1 and E_2 is determined from theorem 6 in a previous paper by the author [Comment. Math. Helv. 22, 168–199 (1949); these Rev. 10, 445], which gives a relation $\zeta(r-1)=\pi^{r-1}\rho\omega$, where ζ denotes the Riemann zeta function and ρ is a certain rational number. Thus the set of numbers $\pi^{-k}\zeta(k)$ ($k=3, 5, 7, \cdots$) is completely characterized by certain properties of modular forms. A special case of the relation mentioned represents $7\pi^{-3}\zeta(3)$ as a scalar product $(\vartheta_{00}^8, \vartheta_{01}^4\vartheta_{10}^4)$, where

$$\vartheta_{00}(\tau)=\sum \exp \pi i m^2\tau, \quad \vartheta_{01}(\tau)=\sum(-1)^m \exp \pi i m^2\tau,$$
$$\vartheta_{10}(\tau)=\sum \exp \pi i(m+\tfrac{1}{2})^2\tau$$

(summation over all rational integers m).

H. D. Kloosterman (Leiden).

Maass, Hans. Automorphe Funktionen von mehreren Veränderlichen und Dirichletsche Reihen. Abh. Math. Sem. Univ. Hamburg 16, 72–100 (1949).

The automorphic functions which the author considers are solutions of the differential equation

$$(1) \qquad \left[x_k^{k+1}\sum_{j=0}^{k}\frac{\partial}{\partial x_j}\left(x_k^{1-k}\frac{\partial}{\partial x_j}\right)+r^2+\tfrac{1}{4}k^2\right]f=0.$$

This equation is a $(k+1)$-dimensional analogue of the two-dimensional equation

$$\left[y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)+\lambda^2\right]f(\tau)=0 \qquad (\tau=x+iy).$$

The latter is invariant under the hyperbolic group $\tau\rightarrow(\alpha\tau+\beta)(\gamma\tau+\delta)^{-1}$ ($\alpha, \beta, \gamma, \delta$ real; $\alpha\delta-\beta\gamma=1$). According to Vahlen [Math. Ann. 55, 585–593 (1902)] the $(k+1)$-dimensional hyperbolic group can be represented as

$$x\rightarrow(\alpha x+\beta)(\gamma x+\delta)^{-1},$$

where x is a vector $x_0+x_1i_1+x_2i_2+\cdots+x_ki_k$ in the Clifford algebra C_k of rank 2^k which is generated by units $i_1, i_2, \cdots, i_k$ satisfying $i_p^2+1=0$, $i_pi_q+i_qi_p=0$ ($p, q=1, 2, \cdots, k$; $p\neq q$) and where $\alpha, \beta, \gamma, \delta$ are suitably chosen Clifford numbers from C_{k-1}. Every function $f(x)$ which is a solution of (1), which satisfies certain regularity conditions and is invariant under the translations of a k-dimensional lattice in the space V_{k-1}: $(x_0, x_1, \cdots, x_{k-1})$ has a Fourier expansion

$$f(x)=u(x_k)+\sum a(\beta)x_k^{\frac{1}{2}k}K_{ir}(2\pi|\beta|x_k)\exp 2\pi i\Re(\beta x),$$

where the summation is extended over all vectors $\beta=(\beta_0, \beta_1, \cdots, \beta_{k-1})\neq 0$ of a k-dimensional lattice in V_{k-1}; K_{ir} is a Bessel function; $|\beta|$ is the norm of the vector β; $\Re(\beta x)$ denotes the real part of the element βx of C_k and $u(x)=a_1x^{\frac{1}{2}k+ir}+a_2x^{\frac{1}{2}k-ir}$ if $r\neq 0$, $u(x)=x^{\frac{1}{2}k}(a_1+a_2\log x)$ if $r=0$. The author now proves that (if $k>1$) necessary and sufficient conditions for the validity of $f(x)=f(-x^{-1})$ are that the functions

$$F_n(y, P_n)=u_n(y)+\sum a(\beta)P_n(\beta)y^{\frac{1}{2}k+n}K_{ir}(2\pi|\beta|y)$$

($n=0, 1, 2, \cdots$; $y>0$) satisfy $F_n(y^{-1}, P_n)=(-1)^nF(y, P_n')$. Here $P_n(\beta)=P_n(\beta_0, \beta_1, \cdots, \beta_{k-1})$ is a spherical harmonic of degree n, P_n' is the conjugate spherical harmonic of P_n and $u_n(y)=u(y)$ or $=0$ according as $n=0$ or $n>0$. By means of the Mellin transformation

$$4\int_0^\infty (F_n(y, P_n)-u_n(y))y^{2s-\frac{1}{2}k-n-1}dy$$
$$=\pi^{-2s}\Gamma(s+\tfrac{1}{2}ir)\Gamma(s-\tfrac{1}{2}ir)\varphi(s, P_n)=\xi(s, P_n)$$

the linear set of the functions $f(x)$ mentioned above is proved to be in one-to-one correspondence with the linear set of Dirichlet series $\varphi(s, P_n)=\sum a(\beta)P_n(\beta)|\beta|^{-2s}$ satisfying the functional equation $\xi(\frac{1}{2}k+n-s, P_n)=(-1)^n\xi(s, P_n')$ and in addition certain regularity conditions. A special case is considered in more detail. It deals with zeta-functions of those biquadratic number fields which contain an imaginary quadratic field $R(\sqrt{d})$. The corresponding functions $f(x)$ are proved to be invariant under the principal congruence group of the field $R(\sqrt{d})$, the level (Stufe) of which is a certain ideal of this field.

H. D. Kloosterman (Leiden).

ANALYSIS

Pólya, G. On the harmonic mean of two numbers. Amer. Math. Monthly 57, 26–28 (1950).

The author proves that max $|p-x|/x$, $x\varepsilon(a, b)$ (i.e., the greatest possible relative error committed by taking p instead of x about which we know only that it lies in the interval (a, b)) is minimized by the harmonic mean $p=2ab/(a+b)$ [and not by the arithmetic mean].

J. Aczél (Miskolc).

Beckenbach, E. F. A class of mean value functions. Amer. Math. Monthly 57, 1–6 (1950).

In connection with the result of G. Pólya reviewed above the author mentions that the sum of the squares of the relative errors $\sum_{j=1}^{n}[(p-a_j)/p]^2$ is minimized by the counter-harmonic mean $p=\sum_{j=1}^{n}a_j^2/\sum_{j=1}^{n}a_j$. In this connection the author succeeds in expounding a comprehensive theory of the general counter-harmonic mean $N_t(a)=\sum_{j=1}^{n}a_j^t/\sum_{j=1}^{n}a_j^{t-1}$, in analogy to the theory of the root-mean-power

$$M_t(a)=\left(\sum_{j=1}^{n}a_j^t/n\right)^{1/t}$$

as treated, e.g., by G. H. Hardy, J. E. Littlewood and G. Pólya [Inequalities, Cambridge University Press, 1934, chapters II, III], although the proofs of the corresponding theorems for the counter-harmonic mean are of course considerably harder than of those for the root-mean-power. The main results are:

$$(1)\qquad \lim_{t\to-\infty} N_t(a)=\min(a_1, a_2, \cdots, a_n),$$

$$\lim_{t\to+\infty} N_t(a)=\max(a_1, \cdots, a_n);$$

$$N_t(ka)=kN_t(a);$$

$N_t(a)\geqq M_t(a)$ for $t\geqq1$ and $N_t(a)\leqq M_t(a)$ for $t\leqq1$ (moreover, $N_t(a)\leqq M_{t-1}(a)$ for $t\leqq0$); (2) $N_t(a)$ is a nondecreasing function of t and for $0\leqq t\leqq1$ an increasing function of every a_k $(k=1, 2, \cdots, n)$ (and also for $t<0$ provided $a_k<N_t(a)$ and for $t>1$ provided $a_k>N_t(a)$); (3) for two sets $(a)=(a_1, a_2, \cdots, a_n)$, $(b)=(b_1, b_2, \cdots, b_n)$ the analogue of Minkowski's inequality holds, $N_t(a)+N_t(b)\geqq N_t(a+b)$ for $1\leqq t\leqq2$ and $N_t(a)+N_t(b)\leqq N_t(a+b)$ for $0\leqq t\leqq1$; for $t>2$ and $t<0$ the respective inequalities can be stated as remaining true only if $b_1=b_2=\cdots=b_n$. The cases of equality are also discussed in all these theorems. The same results hold for weighted counter-harmonic means, for infinite series and for the corresponding integral mean.

By means of this paper the counter-harmonic mean has become changed from a dreadful habitual counter-example in the theory of mean values into an object of systematic research.

J. Aczél (Miskolc).

Karlin, S., and Shapley, L. S. Geometry of reduced moment spaces. Proc. Nat. Acad. Sci. U. S. A. 35, 673–677 (1949).

The authors state a number of theorems, proofs of which are to be published later. These theorems concern the reduced moment space D_n defined in the n-dimensional Euclidean space as the locus of the point $\mu=(\mu_1, \cdots, \mu_n)$, $\mu_k=\mu_k(\phi)=\int_{-0}^{1}t^k d\phi(t)$, as $\phi(t)$ varies over the class of distribution functions $\phi(t)$ which on $0\leqq t\leqq1$ are real, continuous to the right, and nondecreasing, and for which $\phi(-0)=0$, $\phi(1)=1$. The locus of this point is a twisted curve C_n when the $\phi(t)$ are restricted to the functions I_s constant except for a saltus at $t=s$. The space D_n is closed and convex and, when $n\geqq2$, has C_n as its set of extreme points. The first theorem states that every point x of the boundary of D_n corresponds to a unique $\phi(t)$ with moments $\mu_k(\phi)=x_k$ and has a unique representation $x_k=\int_{-0}^{1}\mu_k(I_t)d\phi(t)$ as a convex combination of extreme points. The number of extreme points involved in the convex representation is expressed in terms of the dimensions of the intersection space of all the supporting planes of D_n at x. Similar ideas are applied to the convex closed n-dimensional space P_n of the coefficients of all real nth degree polynomials $P(t)$ which are non-negative on $0\leqq t\leqq1$ and for which $\int_0^1P(t)dt=1$. If the normalizing conditions on both the $\phi(t)$ and $P(t)$ are dropped, the corresponding $(n+1)$-dimensional spaces are convex cones C and C^* which are conjugates in the sense that C^* is the set of points y for which $\sum_{i=1}^{n}x_iy_i\geqq0$ when x varies over C. There is a duality between the coordinates of a boundary point of either cone and the coefficients of the supporting plane of the other cone. This duality, valuable in the theory of "two-person, zero-sum" games, leads to some well-known results on the moments of distributions and provides a geometric approach to the theory of orthogonal polynomials.

M. Marden (Milwaukee, Wis.).

Timan, A. F. Quasi-smooth functions. Doklady Akad. Nauk SSSR (N.S.) 70, 961–963 (1950). (Russian)

Let $\Lambda^*(a, b, M)$ be the class of continuous functions $f(x)$, $a\leqq x\leqq b$, satisfying

$$(*)\qquad |f(x_1)-2f(\tfrac{1}{2}(x_1+x_2))+f(x_2)|\leqq M|x_1-x_2|$$

for all x_1, x_2 in $[a, b]$. It is known that if $f(x)$ is periodic, of period $b-a$, and if (*) holds for all x_1, x_2, then the modulus of continuity $\omega(f, h)$ of f satisfies an inequality $\omega(f, h)\leqq CMh\log(1/h)$ $(0<h\leqq\frac{1}{2})$, where C is a constant independent of f [see A. Zygmund, Duke Math. J. 12, 47–76 (1945); these Rev. 7, 60]. In the present note the following result is stated without proof. If $f(x)$ (not necessarily periodic) is continuous in $[a, b]$ and satisfies (*), then, for every $0\leqq h\leqq(b-a)/4$,

$$(\ddagger)\qquad \omega(f, h)\leqq(M/\log 2)h\log(1/h)+Ah,$$

where A is a certain constant depending on f and $b-a$. The factor $M/\log 2$ here is best possible, and there are functions f for which inequality ($\ddagger$) becomes an asymptotic equality.

A. Zygmund (Chicago, Ill.).

***Mandelbrojt, S. Une inégalité sur les séries asymptotiques.** Analyse Harmonique, Colloques Internationaux du Centre National de la Recherche Scientifique, no. 15, pp. 85–91. Centre National de la Recherche Scientifique, Paris, 1949. 600 francs.

Announcement of results connected with the author's "fundamental inequality." Full statements and proofs have been published elsewhere [Ann. Sci. École Norm. Sup. (3) 43, 351–378 (1947); these Rev. 9, 229; see also Bull. Amer. Math. Soc. 54, 239–248 (1948); these Rev. 9, 416].

W. H. J. Fuchs (Liverpool).

Calculus

✶Narayan, Shanti. A Course of Mathematical Analysis. 2d ed. S. Chand & Co., Delhi, 1949. iv+304 pp. Rs. 15/-.

This is a rigorous second course in the calculus of functions of one or two real variables, which can serve as a foundation for more advanced work. Most of the usual topics are included, starting with real numbers and ending with Fourier series. The treatment is careful throughout, and complete from the arithmetic point of view, but with very limited reference to any geometric motivation of the argument.
P. Franklin (Cambridge, Mass.).

✶Ringleb, F. Mathematische Formelsammlung. 5th ed. Sammlung Göschen Band 51. Walter de Gruyter & Co., Berlin, 1949. 274 pp. 2.40 DM.

"Fünfte, verbesserte Auflage." The first three editions appeared in 1928, 1931, 1936.

Pflanz, E. Allgemeine Differenzenausdrücke für die Ableitungen einer Funktion $y(x)$. Z. Angew. Math. Mech. **29**, 379–381 (1949).

A finite expression and the error term are given for the mth derivative $y^{(m)}(x_0)$ of a function $y(x)$ at a point x_0.
E. Frank (Chicago, Ill.).

Bononcini, Vittorio Emanuele. Interpretazione geometrica dei segni delle derivate successive di una funzione $y=f(x)$. Boll. Un. Mat. Ital. (3) **4**, 267–269 (1949).

Theory of Sets, Theory of Functions of Real Variables

✶Sierpiński, Wacław. Leçons sur les nombres transfinis. Gauthier-Villars, Paris, 1950. vi+240 pp.

"Nouveau tirage" of a work first published in 1928.

We have old edition.

✶Borel, Émile. Leçons sur la théorie des fonctions. (Principes de la théorie des ensembles en vue des applications à la théorie des fonctions). 4th ed. Gauthier-Villars, Paris, 1950. xiii+295 pp. 1200 francs.

The third edition appeared in 1928. This edition has a new preface, some new remarks added as footnotes, and a 5-page appendix, "L'axiome de choix et les définitions asymptotiques," reprinted from C. R. Acad. Sci. Paris **224**, 1537–1538, 1597–1599 (1947); these Rev. **9**, 1.

Levi, F. W. On frequencies and semi-continuous functions. Canadian J. Math. **2**, 32–43 (1950).

Let S be a space in which a family of subsets, called open sets, is distinguished which satisfy the following condition L: if A is the join of a system of open sets, then there exist countably many sets of this system such that A is the join of them. A property F, which for a subset of S either holds or does not hold, is called by the author a "frequency" when it satisfies the following three conditions: (1) F holds in the join of countably many sets A_n if and only if F holds in at least one A_n; (2) F does not hold in the empty set; (3) F holds in S. Examples of such "frequencies" are: F_0 = the property of not being empty; F_1 = the property of not being countable; and, if a regular measure function is defined in S, F_G = the property of having positive measure. If the set M has the frequency F, then a property of a set A for which AM has the frequency F is called a frequency $F(M)$. If every open set which contains a point x has the frequency $F(M)$, then the author says: M has the frequency F at x.

Now let S be a topological space with the second axiom of countability (hence the condition L is satisfied). Then the author defines $f_1(x)$ as the "upper limiting function mod F" of $f(x)$ and $f_2(x)$ as the "lower limiting function mod F" of $f(x)$ in the following way: $f_1(x)$ is the least upper bound of the values k such that for every $\epsilon>0$ the points x' with $f(x')>k-\epsilon$ have the frequency F at x; $f_2(x)$ is the greatest lower bound of the values g such that for every $\epsilon>0$ the points x'' with $f(x'')<g+\epsilon$ have the frequency F at x. The function $f(x)$ for which $f(x)=f_1(x)$ (or $f(x)=f_2(x)$) are called by the author "semicontinuous mod F above (or below)." (Ordinary semicontinuity corresponds to $F=F_0$.) These generalized semicontinuous functions are then discussed in detail by the author. Hereby the notion "nearly every," in the sense of "every except a set which has not the frequency F," plays an essential role.
A. Rosenthal (Lafayette, Ind.).

Halmos, Paul R., and Vaughan, Herbert E. The marriage problem. Amer. J. Math. **72**, 214–215 (1950).

Soient E et F deux ensembles, et pour chaque $x\epsilon E$, soit $\Phi(x)$ une partie finie de F; le "problème des mariages" consiste à trouver à quelle condition il existe une application biunivoque f de E dans F telle que $f(x)\epsilon\Phi(x)$ pour tout $x\epsilon E$. La solution en a été donnée par P. Hall lorsque E est fini, et par Everett et Whaples [même J. **71**, 287–293 (1949); ces Rev. **10**, 517] dans le cas général. Les auteurs montrent que le cas où E est infini se ramène au cas où E est fini par application du théorème de Tychonoff; pour le cas où E est fini, ils donnent du résultat de P. Hall une démonstration plus simple, procédant par récurrence.
J. Dieudonné (Nancy).

de Bruijn, N. G., and Erdös, P. Sequences of points on a circle. Nederl. Akad. Wetensch., Proc. **52**, 14–17 = Indagationes Math. **11**, 46–49 (1949).

Let $a=\{a_n\}$ be a sequence of points on a circle of radius $1/(2\pi)$. The first n points divide the circle into n arcs: the greatest and least values of the length of r consecutive arcs are written $M_n^r(a)$, $m_n^r(a)$. Defining

$$\begin{aligned} \Lambda_r(a)&=\limsup nM_n^r(a), & \Lambda_r&=\inf \Lambda_r(a),\\ \lambda_r(a)&=\liminf nm_n^r(a), & \lambda_r&=\sup \lambda_r(a),\\ \mu_r(a)&=\limsup M_n^r(a)/m_n^r(a), & \mu_r&=\inf \mu_r(a), \end{aligned}$$

the authors evaluate $(\Lambda_1, \lambda_1, \mu_1)$, obtain bounds for $(\Lambda_r, \lambda_r, \mu_r)$ and conjecture that $r(\Lambda_r-1)$, $r(\mu_r-1)$, $r(1-\lambda_r)$ all become infinite with r. The last conjecture would imply van Aardenne-Ehrenfest's theorem [same Proc. **48**, 266–271 = Indagationes Math. **7**, 71–76 (1945); these Rev. **7**, 376].
H. D. Ursell (Leeds).

Kuipers, L., and Meulenbeld, B. Asymptotic C-distribution. I. Nederl. Akad. Wetensch., Proc. **52**, 1151–1157 = Indagationes Math. **11**, 425–431 (1949).

Kuipers, L., and Meulenbeld, B. Asymptotic C-distribution. II. Nederl. Akad. Wetensch., Proc. **52**, 1158–1163 = Indagationes Math. **11**, 432–437 (1949).

A measurable function $f(t)$, $0\leqq t<\infty$, is said to be C^1 uniformly distributed if, for any $0\leqq\alpha\leqq\beta\leqq 1$,

$$\lim_{T\to\infty} T^{-1}\int_0^T \theta(\alpha,\beta; f(t))dt=\beta-\alpha,$$

where $\theta(\alpha, \beta; f(t)) = 1$ if $\alpha \leqq f(t) < \beta \pmod 1$ and $\theta(\alpha, \beta; f(t)) = 0$ otherwise. This is clearly analogous to the definition of a uniformly distributed sequence. One obtains C^{II} uniform distribution if the interval (α, β) is replaced by any countable union of disjoint intervals, and C^{III} uniform distribution if it is replaced by any measurable set. The authors prove that there exists a function $f(t)$ which is C^{I} uniformly distributed but which is not C^{II} uniformly distributed, but that C^{II} and C^{III} are equivalent. They also give various necessary and sufficient conditions for a function $f(t)$ to be C^{I} or C^{II} uniformly distributed, which are applied to various classes of functions. *P. Erdös.*

Kuipers, L. On real periodic functions and functions with periodic derivatives. Nederl. Akad. Wetensch., Proc. **53**, 226–232 = Indagationes Math. **12**, 34–40 (1950).

Let $t \geqq 0$, $f(t)$ differentiable and assume that $f'(t)$ has period p. The author proves that if $f(t+p) - f(t) = \tau$ is irrational then $f(t)$ is C^{I}-uniformly distributed (mod 1) [for the definition of C^{I}-uniform distribution see the preceding review]. If τ is rational $f(t)$ is C^{I}-uniformly distributed (mod 1) if and only if $\int_0^p e^{2\pi i h f(t)} dt = 0$ for any integer $h \neq 0$ with $h\tau$ an integer. The author also proves that if $\pi\alpha$ is irrational, $|\beta| < \alpha$, then $\alpha t + \beta \sin t$ is C^{II} uniformly distributed (mod 1). Finally he proves that if $f''(t)$ is periodic and $f'(t)$ is not periodic, then $f(t)$ is C^{I} uniformly distributed (mod 1). *P. Erdös.*

Nemyckiĭ, V. V. On the definition of an abstract integral. Učenye Zapiski Moskov. Gos. Univ. **135**, Matematika, Tom II, 10–22 (1948). (Russian)

The author considers an integral as a positive normalized linear functional on a certain class of functions and proves some elementary existence and uniqueness theorems. The chief purpose of the work appears to be to present a theory which includes both Riemann and Lebesgue integration as special cases. The paper contains nothing new.

P. R. Halmos (Chicago, Ill.).

Silipigni, L. Teoremi della media. Boll. Accad. Gioenia Sci. Nat. Catania (4) **3**, 175–182 (1949).

This is an expository article on the second mean value theorem for integrals. The author concludes with a statement in terms of Stieltjes integrals $\int f(x) d\alpha(x)$, but only when α is differentiable. [For the general case of Stieltjes integrals, cf. Boas and Widder, Trans. Amer. Math. Soc. **45**, 1–72 (1939), p. 6.] *R. P. Boas, Jr.*

Koval', P. I. On Stieltjes integrals. Učenye Zapiski Moskov. Gos. Univ. **135**, Matematika, Tom II, 152–166 (1948). (Russian)

The author considers the integral $\int_\delta f(x) F(d\Pi)$, called the Stieltjes-Darboux or (D) integral, and defined as follows. Let f be a single-valued bounded point-function on an interval δ and let Π be a system of partitions P of δ into partial intervals, non-overlapping except for end-points, where interval means a set of any one of the forms $a < x < b$, $a < x \leqq b$, $a \leqq x < b$, $a \leqq x \leqq b$, $a = x = b$. The system Π is assumed to contain partitions of arbitrarily small norm and to contain the intersection-partition of every pair of partitions in it. Let F be an interval-function additive and of limited variation relative to Π. If F is also nonnegative, set $\underline{S} = \sup_\Pi \sum_P m(f, \delta_i) F(\delta_i)$, $\bar{S} = \inf_\Pi \sum_P M(f, \delta_i) F(\delta_i)$, where $m(f, \delta_i) = \inf f(x)$, $M(f, \delta_i) = \sup f(x)$ for x on δ_i. Then the (D) integral is S if $S = \underline{S} = \bar{S}$. If $F = F_1 - F_2$, where F_1, F_2 are nonnegative and $S_1 = \int_\delta f(x) F_1(d\Pi)$, $S_2 = \int_\delta f(x) F_2(d\Pi)$, then the (D) integral is defined to be $S_1 - S_2$.

Frequent use is made of the notion [credited to V. I. Glivenko] of resolvent sequence of partitions. A point x of δ is a point of discontinuity of F if there is at least one sequence $\delta_1, \delta_2, \cdots$ of intervals of Π such that $\delta_1 \supset \delta_2 \supset \cdots$, lim length $\delta_n = 0$, $\lim F(\delta_n) = 0$, and for every n, x is either in δ_n or is an end-point of it. Then a resolvent sequence for F is a sequence $P_1, P_2, \cdots$ of partitions of Π such that lim norm $P_n = 0$ and for each point x of discontinuity of F there is an n_0 such that for $n > n_0$ and $P \varepsilon \Pi$ there are as many intervals δ' of P_n for which the statement "x is in δ' or is an end-point of it" is true as there are intervals of P for which that statement is true. The following is true: if $P_1, P_2, \cdots$ is a resolvent sequence, then in order that the (D) integral exist with value L it is necessary and sufficient that $\lim_n \sum_{P_n} f(\xi_i^n) F(\xi_i^n) = L$, where ξ_i^n is an arbitrary point of δ_i^n.

The author gives a formal instance of the integral (D) which he calls the Stieltjes-Glivenko integral. The classical Riemann-Stieltjes integral is shown to be an actual instance. The remainder of the paper is devoted to integrability conditions, axiomatic definition of the integral, non-existence of resolvent sequences for improper integrals, calculation of the total variation, and the extension of the results to double integrals. *H. L. Smith* (Baton Rouge, La.).

Koval', P. I. On Stieltjes integrals. Uspehi Matem. Nauk (N.S.) **4**, no. 6(34), 190–193 (1949). (Russian)

Abstract of the paper reviewed above.

H. L. Smith (Baton Rouge, La.).

Koval', P. I. Conditions of convergence for improper Stieltjes integrals. Učenye Zapiski Moskov. Gos. Univ. **135**, Matematika, Tom II, 167–172 (1948). (Russian)

The author investigates the convergence of the improper (D) integral $\int_\sigma f(x) F(d\Pi) = \lim_{\delta \to \sigma} \int_\delta f(x) F(d\Pi)$, where σ is a set of the form $a < x < +\infty$ or $a \leqq x < +\infty$, $\int_\delta f(x) F(d\Pi)$ is a (D) integral [see the preceding reviews] over a finite interval whose left end-point is the same as the left end-point of σ and this point belongs to both σ and δ or to neither of them. The uniform convergence of $\int_\sigma f(x, \alpha) F(d\Pi)$ is also considered. *H. L. Smith* (Baton Rouge, La.).

Stolyarov, N. A. On a generalization of the Stieltjes integral. Doklady Akad. Nauk SSSR (N.S.) **70**, 15–16 (1950). (Russian)

Let f, φ be bounded functions on a real interval (a, b). Let $\omega(\varphi; c_0, \cdots, c_m)$ be defined by the recursion formulas:

$$\omega(\varphi; c_0) = \varphi(c_0),$$
$$\omega(\varphi; c_0, \cdots, c_m) = [\omega(\varphi; c_0, \cdots, c_{m-1}) - \omega(\varphi; c_0, \cdots, c_{m-2}, c_m)]/(c_{m-1} - c_m).$$

Let k be a natural number. Let $a = x_0 < x_1 < \cdots < x_n = b$ be an arbitrary partition of (a, b) with $n > k$ and form the sum

$$S = \sum f(x_i)[\omega(\varphi; x_i, \cdots, x_{i+k-1}) - \omega(\varphi; x_{i-1}, \cdots, x_{i+k-2})],$$

where the summation is from $i = 1$ to $i = n - k + 1$ inclusive. Then if S converges to a limit as the largest of the differences $x_i - x_{i-1}$ converges to 0, this limit is called by the author a generalized Stieltjes integral and is denoted by $\int_a^b f(x)(d^k\varphi/dx^{k-1})$. For $k = 1$ it reduces to the ordinary Stieltjes integral and for $k = 2$ to a notion introduced by Kantorovitch [C. R. (Doklady) Acad. Sci. URSS (N.S.) **5 (1934 IV)**, 417–421]. Among the properties stated by the author is the following: if f is continuous on (a, b) and if φ

has a derivative of order $k-1$ of limited variation on (a, b), then the integral exists. *H. L. Smith* (Baton Rouge, La.).

Hayes, C. A., Jr., and Morse, A. P. Some properties of annular blankets. Proc. Amer. Math. Soc. 1, 107–126 (1950).

The authors continue the investigations by A. P. Morse and others of general covering theorems of Vitali type, using the notation and some results of earlier papers [A. P. Morse, Trans. Amer. Math. Soc. **55**, 205–235 (1944); **61**, 418–442 (1947); these Rev. **5**, 231; **8**, 571; see also A. P. Morse and J. F. Randolph, ibid. **55**, 236–305 (1944); these Rev. **5**, 232]. An "annular blanket" F on a set A in a separable metric space is a family of Borel sets covering A in the Vitali sense, and such that the sets of F are approximately circles. The object of the paper is to connect the covering properties of such blankets with the ability to differentiate with respect to them. A necessary and sufficient condition is obtained for F and all its sub-blankets to have the Vitali covering property (for a given measure ψ), the condition being expressed in terms of the differentiability of ψ with respect to the blanket F. It is also shown that there exists (in the Euclidean plane) a blanket, whose sets are closed, nearly circular and "star-shaped" about a central point, with respect to which differentiation is impossible for a certain measure. *U. S. Haslam-Jones* (Oxford).

Segal, I. E. Invariant measures on locally compact spaces. J. Indian Math. Soc. (N.S.) 13, 105–130 (1949).

In this very detailed article of partly expository character, a group G of equicontinuous homeomorphisms of a uniform locally compact Hausdorff space M is studied. The existence of a regular σ-finite measure invariant under G and defined on the σ-field generated by the closed compact subsets of M is proved by combining Kodaira's proof of the existence of Haar's measure [Proc. Phys.-Math. Soc. Japan (3) **23**, 67–119 (1941); these Rev. **2**, 317] with an additional argument. This measure is shown to be unique if and only if there is a point in M whose orbit is dense. Moreover, a sufficient condition for the existence of convolution integrals is established and every linear nonnegative functional, defined on the family of all nonnegative continuous functions which vanish outside compact subsets of M, is shown to be the integral over M with respect to its Radon measure.
H. M. Schaerf (Berkeley, Calif.).

Schaerf, Henry M. Sur l'unicité des mesures invariantes. C. R. Acad. Sci. Paris 229, 1053–1055 (1949); errata, 230, 795 (1950).

Let $\mathbf{T}$ be a family of subsets of a space E on which there is defined a relation $A\sim B$ with the following property: (1) $A\sim B$ and $A=\bigcup_{i=1}^{\infty}A_i$ (disjoint; $A, A_i\epsilon\mathbf{T}$, $i=1, 2, \cdots$) imply the existence of $B_i\epsilon\mathbf{T}$ such that $B=\bigcup_{i=1}^{\infty}B_i$ (disjoint) and $A_i\sim B_i$, $i=1, 2, \cdots$. A countably additive measure $m(A)$ defined on $\mathbf{T}$ is called invariant if $m(A)=m(B)$ whenever $A\sim B$. An invariant measure $m(A)$ has the intersection property if: (2) $A, B\epsilon\mathbf{T}$, $m(A)>0$, $m(B)>0$ imply the existence of $A', B'\epsilon\mathbf{T}$ such that $A'\subset A$, $B'\subset B$, $A'\sim B'$ and $m(A')=m(B')>0$. In this paper the uniqueness of invariant measure on $\mathbf{T}$ is first proved under the assumption that every invariant measure on $\mathbf{T}$ has this intersection property. This result is then applied to the case when $E=G$ is a group and when $A\sim B$ means congruence by left translation. The intersection property and hence the uniqueness of left invariant measure follows immediately, by using the Fubini theorem, from the following two conditions: (3) for any $A, B\epsilon\mathbf{T}$ there exists a $D\epsilon\mathbf{T}$ such that $D\supset AB^{-1}$, (4) $(x, y)\rightarrow(yx, y)$ is $\mathbf{T}\times\mathbf{T}$-measurable in the sense of A. Weil [L'intégration dans les groupes topologiques et ses applications, Actual. Sci. Ind., no. 869, Hermann, Paris, 1940; these Rev. **3**, 198]. *S. Kakutani.*

Schaerf, Henry M. Sur l'unicité de la mesure de Haar. C. R. Acad. Sci. Paris 229, 1112–1113 (1949).

The uniqueness of (left-invariant) Haar measure defined on a (left-invariant) σ-ring $\mathbf{T}$ of subsets of a locally compact, not necessarily separable nor commutative, topological group G is proved when $\mathbf{T}$ is contained in the σ-ring $\mathbf{S}$ of all Borel subsets of G ($\mathbf{S}$ is the σ-ring generated by the family of all compact subsets of G) and contains the σ-ring $\mathbf{S}_0$ of all Baire subsets of G ($\mathbf{S}_0$ is the σ-ring generated by the family of all compact G_δ-sets of G). The proof is based on a result of the author [cf. the preceding review] and a result of the reviewer and Kodaira that a (left-invariant) Borel measure defined on $\mathbf{S}$ and its Baire restriction to $\mathbf{S}_0$ have the same completion [Proc. Imp. Acad. Tokyo **20**, 444–450 (1944); these Rev. **7**, 279]. *S. Kakutani.*

Theory of Functions of Complex Variables

Lekkerkerker, C. G. On power series with integral coefficients. II. Nederl. Akad. Wetensch., Proc. 52, 1164–1174 = Indagationes Math. 11, 438–448 (1949).

This paper is a continuation of another by the same author [same Proc. **52**, 740–746 = Indagationes Math. **11**, 270–276 (1949); these Rev. **11**, 338] who now obtains the following generalizations of an earlier theorem. Let $p_1, \cdots, p_s$ be different complex numbers in the domain $0<|p_\sigma|<1$, $\Re p_\sigma\geqq 0$ $(\sigma=1, 2, \cdots, s)$ and let $r_1, \cdots, r_s$ be s positive integers. Given the complex numbers $\eta_{\sigma\rho}$, $(\rho=1, \cdots, r_\sigma; \sigma=1, \cdots, s)$ there exists a power series $f(z)=\sum_{n=0}^{\infty}a_nz^n$ with bounded integral coefficients a_n, so that $f^{(\rho-1)}(p_\sigma)=\eta_{\sigma\rho}$ for every pair σ, ρ. Let $\{A_n\}$ be a sequence of sets of real numbers and $\{g_n\}$ a sequence of positive numbers so that each closed interval of length g_n contains at least one number of A_n for every n. Let h_m denote $\max(g_{(m-1)r+1}, g_{(m-1)r+2}, \cdots, g_{mr})$. Let $G=\liminf_{n\to\infty}(g_n)^{-1/n}$ be positive or infinite. If $p_1, \cdots, p_r$; $\eta_1, \cdots, \eta_r$ are complex numbers such that (1) $p_1, \cdots, p_r$ are different, (2) for each ρ the complex conjugate of p_ρ is p_τ, a member of the set (1) and then $\bar{\eta}_\rho=\eta_\tau$, (3) $0<|p_\rho|<G$, $\rho=1, 2, \cdots, r$, then there exist a power series $f(z)$ and a positive number Q, which depends only on $p_1, \cdots, p_r$, so that (a) a_n belongs to A_n for every n, (b) $|a_{mr+\rho}|\leqq Qh_m+\frac{1}{2}h_{m+1}$ $(\rho=1, \cdots, r;\ m=1, 2, \cdots)$, (c) $f(p_\rho)=\eta_\rho$ $(\rho=1, \cdots, r)$. These results were obtained by a study of determinants whose elements are of the form $(a+r)_{r_\sigma-1}\cdot p_\sigma^{\,r}$, where $(a+r)_t$ denotes $(a+r)(a+r-1)\cdots(a+r-t+1)$.
M. S. Robertson (New Brunswick, N. J.).

***Gattegno, C., et Ostrowski, A. Représentation conforme à la frontière; domaines généraux.** Mémor. Sci. Math., no. 109. Gauthier-Villars, Paris, 1949. 60 pp.

This is a summary of the progress made in the study of the behavior of the boundary of a domain under conformal mapping. It contains the definitions of the various terms connected with the mapping of the boundary, the formulation of the problems, and the statements of the related theorems; no proofs, however, are included. This and the book reviewed below serve as a rather comprehensive bibli-

ography of the subject of the mapping of the boundary. The first chapter is devoted to the classification of boundary points and the definition of prime ends. Chapter two presents the Schwarz reflection principle and several limit theorems like the Julia-Carathéodory theorem. The correspondence of the boundary points of simply connected domains under conformal mapping comes in chapter three along with the theory of variable domains. The more precise results derived by using harmonic measure and the theory of majorants are given in chapter four. *G. Springer.*

✱Gattegno, C., et Ostrowski, A. Représentation conforme à la frontière; domaines particuliers. Mémor. Sci. Math., no. 110. Gauthier-Villars, Paris, 1949. 56 pp.

[Cf. the preceding review.] This book deals with domains whose boundaries satisfy various regularity conditions. As in the first book, definitions, problems, and results, with adequate references to the sources of the material, are presented, but no proofs are included. The first chapter deals with the measure of the accessibility of a Jordan arc and the angular accessibility of boundary points, along with various definitions of tangents. In chapter two, the authors discuss the proportionality of angles at a boundary point and present the distortion theorem for the boundary and its consequences. Also included are the theorems on the folding of a domain. The question of the angular absolute conformality of a conformal transformation at a boundary point is discussed in chapter three. Here conditions are presented under which a domain will have an angular coefficient of conformality at infinity, making use of the notions of domains of comparison. Chapter four (five) deals with the relations between the properties of the boundary and the behavior of the first (higher) derivatives of the function mapping the given domain on the unit circle. Here are discussed those results which depend upon the existence of tangents at points of the boundary and the curvature of the boundary. *G. Springer* (Cambridge, Mass.).

Parreau, Michel. Comportement à la frontière de la fonction de Green d'une surface de Riemann. C. R. Acad. Sci. Paris 230, 709–711 (1950).

Riemann surfaces S possessing Green's functions are considered. The following theorem is established. Let $g(p, a)$ denote the Green's function for a surface S with pole at a, $0<\lambda<+\infty$, $C_{a,\lambda}=[g(p, a)=\lambda]$, $D_{a,\lambda}=[g(p, a)>\lambda]$. If u is harmonic and bounded in $D_{a,\lambda}$ and satisfies, for all $q\epsilon C_{a,\lambda}$, $\liminf_{p\to q} u\geqq 0$, then $u\geqq 0$ in $D_{a,\lambda}$. [The author's statement of his result has been corrected here.] Related results are given. *M. Heins* (Providence, R. I.).

Nehari, Zeev. The radius of univalence of an analytic function. Amer. J. Math. 71, 845–852 (1949).

Let S denote the family of functions $f(z)$ for which the function $\log [f(z)/z]$ is regular and single-valued in a given bounded domain D of finite connectivity containing the point $z=0$. If $f\epsilon S$ and if M is the least upper bound of $|\log [f(z)/z]|$ in D, the author shows that f is schlicht in the largest circle about $z=0$ all of whose points satisfy $|z|k(z, \bar z)\leqq 2\pi/M$. Here $k(z, \xi)$ is the Szegö kernel function [Math. Z. 9, 218–270 (1921)]. If N is a positive number such that $e^{-N}<|f(z)/z|<e^N$, it is shown that f is schlicht in the largest circle about the origin all of whose points satisfy $|z|k(z, \bar z)\leqq \pi^2/(2N)$. Finally, let T be the family of functions for which $\log f'(z)$ is regular and single-valued in D. The following two results are proved for functions of T. Let $|\log f'(z)|\leqq M$ and let r be the radius of the largest circle about the origin all of whose points satisfy $|z|k(z, \bar z)\leqq 2\pi/M$; then the circle $|z|<r$ is mapped by f onto a convex schlicht domain. Let $e^{-N}\leqq|f'(z)|\leqq e^N$ or $|\arg f'(z)|\leqq N$; if r denotes the radius of the largest circle about the origin all of whose points satisfy $|z|k(z, \bar z)\leqq \frac{1}{2}\pi^2/N$, then $|z|<r$ is mapped by f onto a convex schlicht domain. All bounds are best possible. *D. C. Spencer.*

Royden, H. L. The coefficient problem for bounded schlicht functions. Proc. Nat. Acad. Sci. U. S. A. 35, 657–662 (1949).

Let $\mathcal{S}$ denote the class of functions $f(z)=z+\sum_2^\infty a_n z^n$, regular and schlicht for $|z|<1$, and $\mathcal{BS}$ the class of functions $f(z)=\sum_1^\infty b_n z^n$, $b_1>0$, regular, schlicht and bounded by unity in $|z|<1$. The set of points $(a_2, \cdots, a_n)$ which are associated with elements $f(z)$ of $\mathcal{S}$ forms a closed domain V_n in $(2n-2)$-space, and the set $(b_1, b_2, \cdots, b_n)$ forms a region $\mathcal{V}_n$ in $(2n-1)$-space which is closed if the origin is added. The author shows how $\mathcal{V}_n$ may be constructed from V_n. In an open set containing $\mathcal{V}_n+0$ let a real-valued function F have a continuous and non-vanishing gradient. Then every function $f(z)$ of $\mathcal{BS}$ belonging to the point $P(b_1, b_2, \cdots, b_n)$ where F attains its maximum in $\mathcal{V}_n$ satisfies

$$(1) \qquad \left(\frac{z}{w}\frac{dw}{dz}\right)^2 R(w)=Q(z)$$

[see Charzyński, Colloquium Math. 1, 168–170 (1948)], where R and Q are certain rational functions depending upon the point P and upon F, and which are real and non-negative on the unit circumference with at least one zero there. Equations (1) are called $\mathfrak{D}_n$-equations and the associated solutions $w=f(z)$ $(0<b_1<1)$ $\mathfrak{D}_n$-functions. Letting $\zeta=\int\{Q(z)\}^{\frac{1}{2}}z^{-1}dz$, we say that the set of arcs $\Re(\zeta)=$constant having an end point at a zero of $Q(z)$ forms a structure Γ_z, and a structure Γ_w is similarly defined with $R(w)$.

If $w=f(z)$ is a $\mathfrak{D}_n$-function, then it is schlicht for $|z|<1$, and maps $|z|<1$ onto the interior of the unit circle minus a sub-continuum of Γ_w containing $|w|=1$. Moreover, $f(z)$ is regular on $|z|=1$ except for a finite number of points. Every $\mathfrak{D}_n$-function belongs to some boundary point of $\mathcal{V}_n$, and conversely to any boundary point of $\mathcal{V}_n$ (except the origin) there belongs exactly one $\mathfrak{D}_n$-function. The author obtains a partial differential equation for the boundary of $\mathcal{V}_n$. The characteristic equations are equivalent to Löwner's differential equation for slit mappings. A solution of (1) for the case $n=2$ is obtained and then $\mathcal{V}_2$ is found to have $|b_2|=2b_1(1-b_1)$, as was found by G. Pick [Akad. Wiss. Wien, S.-B. IIa. 126, 247–263 (1917)].

M. S. Robertson (New Brunswick, N. J.).

Goluzin, G. M. Some estimates for bounded functions. Mat. Sbornik N.S. 26(68), 7–18 (1950). (Russian)

The author considers the following classes of functions $f(z)=\sum_{n=0}^\infty c_n z^n$ regular in $|z|<1$: B such that $|f|\leqq 1$; R such that $\Re f(z)\geqq 0$; B_G such that $f(z)$ takes only values in the closed convex region G. He gives the following theorems, in which γ_n are complex numbers, $m\geqq 0$, $\sum_{n=m}^\infty|\gamma_n|<\infty$. (1) The estimate $|\sum_{n=0}^\infty \gamma_n c_n|\leqq\gamma_0$ holds for all $f\epsilon B$ if and only if (*) $\Re\{\gamma_0^{-1}\sum_{n=0}^\infty\gamma_n\zeta^n\}\geqq\frac{1}{2}$ for $|\zeta|=1$; the estimate $|\gamma_m^{-1}\sum_{n=m}^\infty\gamma_n c_n+\bar\gamma_m^{-1}\sum_{n=0}^{m-1}\bar\gamma_{2m-n}c_n|\leqq 1$ holds for all $f\epsilon B$ if and only if $\Re\{\gamma_m^{-1}\sum_{n=m}^\infty\gamma_n\zeta^{n-m}\}\geqq\frac{1}{2}$ for $|\zeta|=1$. (2) The estimate $\Re\{\gamma_0^{-1}\sum_{n=0}^\infty\gamma_n\zeta^n\}\geqq 0$ holds for all $f\epsilon R$ if and only if (*) is true. (3) If $\limsup|\gamma_n|^{1/n}=1$, then $\gamma_0^{-1}\sum_{n=0}^\infty\gamma_n c_n\epsilon G$ whenever the series converges if and only if (*) holds for

$|\zeta|<1$. (4) Let $\gamma_0^{(1)}\neq 0$, $\gamma_0^{(2)}=0$, $\sum_{n=0}^{\infty}(|\gamma_n^{(1)}|+|\gamma_n^{(2)}|)<\infty$. The estimate $|\sum_{n=0}^{\infty}\gamma_n^{(1)}c_n|+|\sum_{n=0}^{\infty}\gamma_n^{(2)}c_n|\leqq|\gamma_0^{(1)}|$ holds if and only if

$$\Re\{(1/\gamma_0^{(1)})\sum_{n=0}^{\infty}\gamma_n^{(1)}\zeta^n\}-\tfrac{1}{2}\geqq|(1/\gamma_0^{(1)})\sum_{n=0}^{\infty}\gamma_n^{(2)}\zeta^n|$$

for $|\zeta|=1$. The proofs are by "Fourier" methods based on the integral formula for the coefficients.

As applications, the author deduces theorems (5)–(10), as follows. (5) If $f\varepsilon B$, then in $|z|\leqq 2^{1/(m+1)}-1$ we have $|f^{(m)}(z)|\leqq m!$ if $c_0=\cdots=c_{m-1}=0$ and

$$|f^{(m)}(z)+\sum_{n=1}^{\infty}(n+1)(n+2)\cdots(n+m)c_{m-n}z^n|\leqq m!$$

in any case. (6) For $|z|\leqq\rho_m$ (with a specific ρ_m) the partial sums of $f\varepsilon B$, of order m, are at most 1 in absolute value; those of $f\varepsilon R$ are nonnegative; those of $f\varepsilon B_G$ belong to G. (7) Let $\rho_1=\frac{1}{3}$, ρ_m the positive root of $4\rho^m+\rho-1=0$. If $s_m(z)$ is the mth partial sum of $f(z)$ and $r_m(z)=f(z)-s_m(z)$, then $|s_m(z)|+|r_m(z)|\leqq 1$ for $|z|<\rho_m$ if $f\varepsilon B$. (8) If $f\varepsilon B$, the estimate $|\bar{z}^m s_m(\bar{z}^{-1})+z^{-m}r_m(z)|\leqq 1$ holds in $|z|<1$. (9) The arithmetic means of $s_n(z)$ are in G for $|z|<1$ if $f\varepsilon B_G$. (10) If $f\varepsilon B$, the sum of the absolute values of the even and odd parts of $f(z)$ does not exceed 1 in $|z|\leqq 2^{\frac{1}{2}}-1$. All these inequalities are sharp, with equality only for $f(z)=c$ or $f(z)=cz^m$, and do not always hold in larger circles than those stated. (11) If $f\varepsilon B$ then $|c_n|\leqq 1-|c_0|^2$.

The author gives no references to the extensive literature of the subject except to a paper of his own and to the source for the first part of (5) for $m=1$ [Dieudonné, Ann. Sci. École Norm. Sup. (3) **48**, 247–358 (1931); Carathéodory, Prakt. Akad. Athēnōn **11**, 276–286 (1936)]. However, a result equivalent to (2) was given by Schur [Arch. Math. Phys. **27**, 126–135 (1918); cf. also Schur and Szegö, S.-B. Preuss. Akad. Wiss. **1925**, 545–570]. Theorem (1) for $m=0$ and the parts of (6) referring to B and R were given by Schur and Szegö [loc. cit.]. Theorem (11) is a well-known consequence of Schwarz's lemma, but apparently the author was interested in showing how it could be obtained by the methods used in this paper. It is also a special case of a theorem of Carlson [Ark. Mat. Astr. Fys. **27A**, no. 1 (1940); these Rev. **2**, 185], which in turn is contained in the following result of Izumi [Tôhoku Math. J. **32**, 303–305 (1930)]: if $f\varepsilon B$, then $(1-\sum_{k=0}^{n}|c_k|^2)^2\geqq\sum_{n+1}^{\infty}|c_k|^2$.

R. P. Boas, Jr. (Providence, R. I.).

Agmon, Shmuel. Sur les séries de Dirichlet. Ann. Sci. École Norm. Sup. (3) 66, 263–310 (1949).

This thesis is concerned with functions represented by a convergent Dirichlet series

$$(*)\qquad f(s)=\sum d_n e^{-\lambda_n s}$$

$(0<h<\lambda_{n+1}-\lambda_n<H)$ in $\sigma>0$. Very general theorems are obtained in the following fields. (1) Fabry gap theorem; (2) Fatou's conjecture on the non-continuability of $\sum\pm d_n e^{-\lambda_n s}$ for almost all choices of the signs; (3) Fabry's condition ($|\arg d_n|\leqq\alpha<\frac{1}{2}\pi$) that $s=0$ should be a singular point of (*); (4) connection between the density of the λ_n and the density of singularities of $f(s)$ on $\sigma=0$. Typical examples of the full statements (with $\lambda_n=n$) have been published previously [C. R. Acad. Sci. Paris **226**, 1673–1674, 1875–1876, 1786–1787 (1948); these Rev. **9**, 576]. All results are deduced by a unified method which is founded on the following theorem 1. Let $y=C(x)$ be the least nonnegative concave envelope of the points $(\lambda_n, \log|d_n|)$, $q_n=\exp(C(\lambda_n))$. Let $f(s)$ be regular in the closed, bounded region D of the s-plane. Write $f_k(s)$ for the kth partial sum of (1). Then, for s in D, $|g_k(s)|=|(f(s)-f_k(s))/q_k e^{-\lambda_k s}|=O(1)$ as $k\to\infty$. The theorem remains valid for a much wider class of majorising sequences $\{q_n\}$, in particular it can be arranged that infinitely often $(\ddagger)$ $q_k\leqq|d_k|$ and that $q_k\to 0$, if $d_k\to 0$. By theorem 1 the $g_k(s)$ form a normal family. Restricting attention to the subset F of this family for whose indices $(\ddagger)$ holds the author proves theorem 2. If $f(s)$ can be continued analytically across $\sigma=0$ and if $g(s)$ is the limit of a convergent sub-sequence of F, then $g(s)$ can be continued analytically throughout the s-plane, except for singularities on $\sigma=0$. The singularities of $g(s)$ are also singularities of $f(s)$. Further detailed discussion of the functions $g(s)$ yields all results. An important rôle is played by the generalisation of a theorem of Cartwright on Laurent series [see Levinson, Gap and Density Theorems, Amer. Math. Soc. Colloquium Publ., v. 26, New York, 1940, theorem 16, p. 21; these Rev. **2**, 180] to Dirichlet series. [Misprint: p. 271, top line, should read $\log(q_n/\Gamma(\lambda_n+1))\cdots$ instead of $\log q_n/\Gamma(\lambda_n+1)\cdots$.]

W. H. J. Fuchs (Liverpool).

Wittich, Hans. Bemerkung zu einer Funktionalgleichung von H. Poincaré. Arch. Math. 2, 90–95 (1950).

Let $p(z)$ and $q(z)$ be polynomials in z, with $p(z)$ nonconstant; let s be a complex number with $|s|>1$. It is shown that if an integral function $f(z)$ is such that $f(sz)=p(z)f(z)+q(z)$, then $f(z)$ satisfies no algebraic differential equation. For the case in which s is an integer not less than 2, it is shown that $f(z)$, as above, may assume integral values at the points s^n, $n=1, 2, \cdots$.

J. F. Ritt (New York, N. Y.).

***Carleson, Lennart. On a Class of Meromorphic Functions and Its Associated Exceptional Sets.** Thesis, University of Uppsala, 1950. 79 pp.

Let $f(z)$ be meromorphic in $|z|<1$. Let $A(r)$ be the area on the Riemann sphere of the image of the circle $|z|\leqq r<1$. The author considers the class T_α of functions $f(z)$ for which

$$T_\alpha[f]=\int_0^1 A(r)(1-r)^{-\alpha}dr<\infty,\qquad 0\leqq\alpha<1,$$

$$T_1[f]=\lim_{r\to 1-0}A(r)<\infty,\qquad \alpha=1.$$

Thus T_0 is the class of functions of bounded characteristic. In chapter I the author defines nullsets E in the sense of the following measures: (i) $L_\alpha(E)$ which is Hausdorff α-dimensional measure if $\alpha>0$ and logarithmic measure if $\alpha=0$; $\dim E\leqq\alpha$ if for every $\epsilon>0$, $L_{\alpha+\epsilon}(E)=0$; (ii) logarithmic α-dimensional measure corresponding essentially to the Hausdorff function $(\log r^{-1})^{-\alpha}$ and (iii) α-dimensional capacity $C_\alpha(E)$, which is positive if it is possible to distribute a mass $\mu(\xi)$ over the set E so that the following is uniformly bounded in z, and is otherwise zero:

$$\int_E|z-\xi|^{-\alpha}d\mu(\xi),\quad \alpha>0;\qquad \int_E\log^+|z-\xi|^{-1}d\mu(\xi),\quad \alpha=0.$$

Thus $C_0(E)$ is equivalent to harmonic measure. For any set E and $\alpha\geqq 0$, $\epsilon>0$, we have

$$L_\alpha(E)=0\Rightarrow C_\alpha(E)=0\Rightarrow L_{\alpha+\epsilon}(E)=0.$$

The following are some of the author's main results. Radial limits $f(e^{i\theta})$ exist for $f(z)\varepsilon T_\alpha$ except for a set E of θ for which $C_{1-\alpha}(E)=0$. If E_a is the set for which the radial limit is a, then $C_{1-\alpha}(E_a)=0$ almost everywhere in a. Values taken frequently by the function $f(z)\varepsilon T_\alpha$ are exceptional in

the following sense. Let $r_\nu(a)$ be the moduli of the zeros of $f(z)-a$ in $|z|<1$ and let S_γ be the set of a for which $\sum[1-r_\nu(a)]^{1-\gamma}$ diverges, $0\leqq\gamma\leqq\alpha$. Then (a) S_0 is empty, (b) dim $S_\gamma\leqq 2\gamma/\alpha$, (c) $L_2(S_\alpha)=0$.

On the other hand a Blaschke product with real positive zeros belongs to any T_α with $\alpha<\frac{1}{2}$, even though $a=0$ may be in every S_γ with $\gamma>0$. By taking a suitable ratio of two such Blaschke products every real a belongs to all the S_γ, whence dim $S_\gamma\geqq 1$. Thus using (b), the ratio of two Blaschke products in T_α with $\alpha>0$ need not be in any positive T_β. The author shows nevertheless that every $f(z)$ in T_α can be expressed as the ratio of two bounded functions belonging to T_β for every $\beta<\alpha$, and conjectures that they need not belong to T_α.

In the last chapter the author obtains some conditions for the function $w(z)$, which maps $|z|<1$ onto the universal covering surface over a domain D in the w-plane, to belong to T_α. Let F be the complement of D. As is well known, $w(z)\epsilon T_0$ if and only if $C_0(F)>0$. It is obvious that $w(z)\epsilon T_1$ if and only if D is simply connected. Even if F consists of a disc and a convergent sequence, $w(z)$ need not belong to any T_α with $\alpha>0$. If $w(z)$ belongs to T_α with $\alpha=\frac{1}{2}$, F is perfect, but if $\alpha<\frac{1}{2}$, F may have infinitely many isolated points. *W. K. Hayman.*

Noshiro, Kiyoshi. Contributions to the theory of the singularities of analytic functions. Jap. J. Math. 19, 299–327 (1948). See the note to p. 241, Tsuji.

L'auteur complète des résultats connus sur le comportement d'une fonction $w=f(z)$ méromorphe, ayant un ensemble de singularités essentielles E borné et de capacité nulle. Il étend la théorie des surfaces de recouvrement d'Ahlfors: l'épuisation de la surface de Riemann R_w (supposée construite sur la sphère S_w projection stéréographique du plan w) est faite au moyen des images F_r données par $w=f(z)$ à partir des ensembles C_r: $U^\mu(z)<\log 2$, $r\rightarrow\infty$; U^μ est le potentiel d'une distribution sur E qui tend vers $+\infty$ en tout point de E; C_r joue le rôle des cercles $|z|<r$ dans la théorie classique. La relation

$$\sum_{j=1}^{q}\delta(D_j)+\sum_{j=1}^{q}\theta(D_j)\leqq 2+\xi$$

est établie; les D_j sont q disques tracés sur S_w; $\delta(D_j)$ est l'indice de défaut, $\theta(D_j)$ celui de ramification et $\xi=\lim\sup_{r\to\infty} n(r)/S(r)$ où $n(r)$ est le nombre des courbes fermées de l'ensemble $U^\mu=\log r$; $S(r)=\pi^{-1}A(r)$ est le recouvrement moyen de le sphère S_w. Si l'on considère une branche $z=z_\nu(w)$ de la fonction inverse sur un voisinage $|w-\omega|<\rho$ d'une singularité essentielle Ω sur R_w, si la branche couvre un domaine Δ_z simplement connexe, alors

$$\sum_{j=1}^{q}\delta(D_j,\Delta z)+\sum_{j=1}^{q}\theta(D_j,\Delta z)\leqq 1.$$

Application à la représentation conforme: soit e un compact de capacité nulle contenu dans le cercle c: $|w|<1$, et soit $D=c-e$; si $\bar{D}$ est la surface de recouvrement universelle de D, et $w=f(z)$ la représentation conforme de $|z|<1$ sur $\bar{D}$, alors si e contient au moins 2 points, l'ensemble des singularités essentielles de $f(z)$ est de mesure nulle mais est nécessairement de capacité positive. En complément à un resultat de Tsuji l'auteur montre que si $w(z)$ est une algébroïde ayant un ensemble E_z de singularités essentielles borné, de capacité nulle, le voisinage Φ_z déterminé par $|w-\omega|<\rho$ sur R_w d'une singularité transcendante Ω sur R_w couvre une infinité de fois toute valeur du cercle $|w-\omega|<\rho$, sauf un ensemble de capacité nulle. Il est à noter que, même dans le cas d'une relation $G(z,w)=0$, définie pour $z\epsilon d$, non nécessairement algébrique en w, mais seulement entière d'ordre de croissance finie en w, on démontre par une méthode simple que l'ensemble E_z des valeurs lacunaires est dénombrable sans point d'accumulation dans d [cf. P. Lelong, C. R. Acad. Sci. Paris 214, 53–54 (1942); Bull. Sci. Math. (2) 66, 103–108, 112–125 (1942); Ann. Sci. École Norm. Sup. (3) 58, 83–177 (1941); ces Rev. 4, 138; 5, 235; 7, 151]. *P. Lelong* (Lille).

Petersson, H. Über Weierstrasspunkte und die expliziten Darstellungen der automorphen Formen von reeller Dimension. Math. Z. 52, 32–59 (1949).

This paper is concerned with automorphic forms of negative dimension and multipliers of modulus one, defined on a group of linear transformations with an orthogonal circle. The central problem treated by the author is the explicit construction (in terms of Poincaré series) of a basis of the forms belonging to a given dimension and group. Criteria for the linear independence of forms and for the completeness of a basis are given and examples are discussed. *Z. Nehari* (St. Louis, Mo.).

Theory of Series

✶Karamata, J. Teorija i Praksa Stieltjesova Integrala. [Theory and Application of the Stieltjes Integral]. Srpska Akademija Nauka, Posebna Izdanja, Kn. 144, Matematički Institut, Kn. 1. Belgrade, 1949. viii+328 pp.

A textbook containing a careful exposition of all details and a wealth of examples and counter-examples. There are four main parts. Some 36 pages are devoted to general properties of functions of bounded variation. Next some 70 pages treat the Stieltjes integral, mean value theorems, etc. The third and main part [about 110 pages] is called "applications." Its chapter headings are, approximately: (1) Functions of sequences of numbers, (2) Applications to infinite series, (3) General summation formulas, (4) Special summation formulas, (5) Dirichlet series, (6) Behavior of Dirichlet series on the boundary of the domain of convergence, (7) Analytic continuation to the left of the abscissa of convergence. The last section [about 95 pages] contains various notes and explanations concerning infinite series, calculus, and inequalities. *W. Feller.*

Olver, F. W. J. Transformation of certain series occurring in aerodynamic interference calculations. Quart. J. Mech. Appl. Math. 2, 452–457 (1949).

Reuter, G. E. H. Note on the preceding paper. Quart. J. Mech. Appl. Math. 2, 457–459 (1949).

A slowly convergent series involving a certain algebraic function is transformed into a rapidly convergent series involving Bessel functions. The first paper uses contour integration and the second uses Poisson's summation formula. *R. P. Boas, Jr.* (Providence, R. I.).

Pucci, Carlo. Un teorema di derivazione per serie con una applicazione alle serie trigonometriche. Boll. Un. Mat. Ital. (3) 4, 270–274 (1949).

If $\{f_n(x)\}$ converges in a set dense in (a,b), and if the functions $f_1^{(r)}(x)$, $f_2^{(r)}(x)$, $\cdots$, $f_n^{(r)}(x)$, $\cdots$ are equicontinu-

ous, then $f_n(x)$ converges uniformly in (a, b) to a limit $f(x)$, and $f^{(p)}(x) = \lim f_n^{(p)}(x)$, uniformly in x, for $p = 1, \cdots, r$.
A. Zygmund (Chicago, Ill.).

Henstock, R. The efficiency of matrices for bounded sequences. J. London Math. Soc. 25, 27–33 (1950).

Let z_n be a given bounded complex sequence. The author defines a set S of sequences of zeros and ones with the following property. If A is a regular matrix method for evaluation of sequences and if each sequence in S is evaluable A, then the sequence z_n is also evaluable A.
R. P. Agnew (Ithaca, N. Y.).

Vernotte, Pierre. L'emploi de la condition de régularité dans la sommation des séries divergentes. Calcul de quelques séries très divergentes. C. R. Acad. Sci. Paris 230, 505–506 (1950).

Using terminology introduced in his book [Théorie et pratique des séries divergentes, Publ. Sci. Tech. Ministère de l'Air, Paris, no. 207 (1947); these Rev. 11, 97] the author suggests methods for obtaining approximations to values of divergent series such as $\sum(-1)^n(n!)^2$. *R. P. Agnew.*

Thron, W. J. Singular points of functions defined by *C*-fractions. Proc. Nat. Acad. Sci. U. S. A. 36, 51–54 (1950).

Let $\{\alpha_n\}_1^\infty$ be a nondecreasing sequence of positive integers such that $\alpha_n \to \infty$ as $n \to \infty$. Let $\{d_n\}_1^\infty$ be a sequence of numbers different from 0 such that $|d_n|^{1/\alpha_n} \to 1$ as $n \to \infty$. The *C*-fraction

$$1 + \mathop{K}_{n=1}^{\infty}\left(\frac{d_n x^{\alpha_n}}{1}\right)$$

converges for $|x| < 1$ to a function $f(x)$ which is analytic except possibly for poles if $|x| < 1$, and has a singularity which is not a pole for a value of x such that $|x| = 1$. Furthermore, if there exists a sequence of positive integers $\{\mu_k\}_1^\infty$ such that $\mu_k \to \infty$ as $k \to \infty$ and, for each positive integer k, μ_k divides α_n for all but a finite number of values of n, then $f(x)$ has the unit circle as natural boundary. [Correction, communicated by the author. Let $\sigma_{2n} = \sum_{k=1}^n \alpha_{2k}$, $\sigma_{2n+1} = \sum_{k=0}^n \alpha_{2k+1}$, $\tau_{2n} = \sum_{k=1}^n \alpha_{2k}$, $\tau_{2n+1} = \sum_{k=1}^n \alpha_{2k+1}$. As the reviewer observed, $s_n = \sigma_n$, $t_n = \tau_n$ does not always hold under the assumptions made in the paper. However these assumptions insure that $s_n \leqq \sigma_n$, $t_n \leqq \tau_n \leqq \sigma_n$. Thus $\sigma_n \geqq \max(s_n, t_n)$. This together with the fact that relations (4) and (5) hold if s_n is replaced by σ_n, allows us to carry through the proof by replacing s_n' by σ_n' in the last 16 lines of the paper.]
H. S. Wall (Austin, Tex.).

Fourier Series and Generalizations, Integral Transforms

***Franklin, Philip. Fourier Methods.** McGraw-Hill Book Company, Inc., New York, N. Y., 1949. x+289 pp. \$4.00.

The titles of the five chapters in this book are Complex quantities and impedance, Fourier series and integrals, Partial differential equations, Boundary value problems, and Laplace transforms and transients. As the title indicates the emphasis in the book is upon methods. Fourier theorems are stated without proof and other parts of the theory are omitted, since it is the purpose of the book to serve as a one-semester introduction to the use of Fourier series and Laplace transforms for students with a working knowledge of elementary calculus. Applications include problems in electric networks, vibrations of mechanical systems and strings, transmission lines, heat conduction and wave guides. A short section on numerical harmonic analysis is also presented. There is a good supply of exercises for the reader. *R. V. Churchill.*

***Jaeger, J. C. An Introduction to the Laplace Transformation with Engineering Applications.** Methuen & Co., Ltd., London; John Wiley & Sons, Inc., New York, N. Y., 1949. viii+132 pp. \$1.50.

This monograph presents a substantial treatment of the portion of the applications and formal theory of the Laplace transformation that is independent of the theory of functions of complex variables. It contains an interesting variety of problems in electric circuit theory, including vacuum tubes, servomechanisms, mechanical analogues to circuit problems, practical methods of determining conditions for stable solutions of circuit problems, and a number of problems concerning transmission lines. The book is divided into four chapters entitled Fundamental theory, Electric circuit theory, Further theorems and their applications, and Partial differential equations. This book should serve well the author's objective of furnishing a good working knowledge of operational methods to students whose mathematical equipment is limited; it should also serve as a valuable reference to those interested in extending their knowledge of applications. A few exercises for the reader are presented.
R. V. Churchill (Ann Arbor, Mich.).

Timan, A. F. On some methods of summation of Fourier series. Izvestiya Akad. Nauk SSSR. Ser. Mat. 14, 85–94 (1950). (Russian)

Let $S_n(x, f)$ be the nth partial sum of a continuous function $f(x)$ of period 2π. A theorem of Rogosinski [Math. Ann. 95, 110–134 (1925)] asserts that if $\alpha_n = \pi k/(2n+1)$, where k is a fixed odd integer and $n = 1, 2, \cdots$, then

$$(*) \qquad \tfrac{1}{2}\{S_n(x+\alpha_n, f) + S_n(x-\alpha_n, f)\} \to f(x)$$

uniformly in x. The result obviously holds if $k = k(n)$ varies with n, provided it is odd and bounded. The main result of the present paper is that a necessary and sufficient condition for a sequence of real numbers α_n ($|\alpha_n| \leqq \pi$) to satisfy (*) uniformly in x for every continuous f of period 2π, is that $\alpha_n = \pi k(n)/(2n+1) + O(1/n \log n)$, where $k(n)$ is odd and bounded. Corresponding results hold for interpolating polynomials and for multiple Fourier series. *A. Zygmund.*

***Beurling, Arne. Sur les spectres des fonctions.** Analyse Harmonique. Colloques Internationaux du Centre National de la Recherche Scientifique, no. 15, pp. 9–29. Centre National de la Recherche Scientifique, Paris, 1949.

ordered

L'auteur définit le "spectre" Λ_f d'une fonction $f(x)$ $(-\infty < x < \infty)$, mesurable et telle que

$$(1) \qquad \int_{-\infty}^{\infty} |f(x)| e^{-\sigma|x|} dx < \infty$$

pour $\sigma > 0$, de la façon suivante: Λ_f consiste des nombres réels λ tels qu'on ait pour tout $\epsilon > 0$

$$\liminf_{\sigma \to +0} \int_{\lambda-\epsilon}^{\lambda+\epsilon} |U_f(\sigma, t)| dt > 0,$$

où

$$U_f(\sigma, t) = \int_{-\infty}^{\infty} f(x) e^{-itx-\sigma|x|} dx;$$

$U_f(\sigma, t)$ est harmonique dans le demi-plan $\sigma>0$. Le spectre est toujours un ensemble fermé. Le problème d'une "synthèse harmonique" est de représenter ou approcher la fonction $f(x)$, dans une certaine topologie, par des sommes ou intégrales étendues à des fonctions de la forme $cx^n e^{i\lambda x}$ où $\lambda \epsilon \Lambda_f$ et n est une entier non-négatif. Pour que cela soit possible, il est tout d'abord nécessaire qu'on possède un "théorème d'unicité" affirmant que $f(x)$ s'annule presque partout si son spectre est vide. La seule condition (1) ne suffit pas pour un tel théorème. Mais si l'on restreint la classe des fonctions envisagées en exigeant que

$$(2) \qquad \int_{-\infty}^{\infty} |f(x)| e^{-\sigma|x|} dx \leqq e^{h(\sigma)} \qquad (\sigma>0)$$

où $h(\sigma)$ est une fonction décroissante pour $\sigma>0$ et telle que $\int_0^1 \log^+ h(\sigma) d\sigma < \infty$, le théorème d'unicité subsiste. [Dans la démonstration, l'auteur fait usage d'un théorème de N. Sjöberg, Comptes Rendus du IX. Congrès des Mathématiciens Scandinaves, Helsingfors, 1938.] De plus, on a pour toute fonction f satisfaisant à (2), l'inégalité (3) $|U_f(\sigma, t)| \leqq H(\sigma, t; \Lambda_f)$ $(\sigma>0)$ où H est une fonction dépendant seulement de $h(\sigma)$ et du spectre Λ_f, et telle que, pour t restant à l'intérieur d'un intervalle complémentaire du spectre, H tend uniformément vers 0 lorsque $\sigma \rightarrow +0$.

La majeure partie de la conférence s'occupe de l'analyse harmonique dans des espaces fonctionnels normés, en particulier dans l'espace de Hilbert E_α des fonctions à carré sommable par rapport à la fonction de poids $p(x) = (1+|x|^\alpha)^{-1}$ $(-\infty < x < \infty)$, α étant un nombre fixe, $0<\alpha\leqq 1$. Pour $f \epsilon E_\alpha$, on a

$$\int_{-\infty}^{\infty} |f(x)| e^{-\sigma|x|} dx \leqq \|f\| (\sigma^{-1} + \sigma^{-1})$$

et

$$|U_f(\sigma, t)| \leqq 3\|f\| (\sigma r^{-2} + \sigma r^{-1}) \qquad (\sigma>0)$$

où $r = r(\sigma, t)$ est la distance du point (σ, t) à l'ensemble Λ_f (cas particulier de (3)). Soit Λ un ensemble fermé donné; la "classe spectrale" E_Λ est définie comme l'ensemble des fonctions $f \epsilon E_\alpha$ pour lesquelles $\Lambda_f \subset \Lambda$. Pour tout tel ensemble Λ, une "mesure spectrale" est définie de la manière suivante: (1) $M(\Lambda)=0$ si E_Λ est vide, (2) $M(\Lambda) = (\inf \|f\|^2)^{-1}$ pour $f \epsilon E_\Lambda$, $f(0)=1$, si Λ est borné et E_Λ n'est pas vide, (3) $M(\Lambda) = \lim_{a\to\infty} M[\Lambda \cdot (-a, a)]$ si Λ n'est pas borné. La fonction $M(\Lambda)$ n'est pas une fonction additive d'ensemble, mais elle reste invariante par rapport aux translations de Λ. Dans le cas limite (exclu) $\alpha=0$, $M(\Lambda)$ se confond avec la mesure linéaire, à un facteur numérique près. L'auteur montre que, dans le cas (2), la borne inférieure $\inf \|f\|^2$ est atteinte par une fonction et une seule et que cette fonction est définie positive. Il s'ensuit que $M(\Lambda)$ est égale, dans ce cas, à la capacité de l'ensemble Λ par rapport au noyau $q(t) = \int_{-\infty}^{\infty} e^{-itx} p(x) dx$, c'est-à-dire, que

$$M(\Lambda)^{-1} = \inf \int\int q(t-\tau) d\mu(t) d\mu(\tau),$$

la borne inférieure étant prise pour toutes les distributions possibles $\mu(t)$ de masses positives sur Λ, de masse totale égale à 1.

Le problème de la "synthèse harmonique" est résolu, pour l'espace E_α, de la manière suivante. A tout $\epsilon>0$ on peut déterminer une fonction complètement additive d'ensemble μ, s'annulant en dehors de l'ensemble Λ_f et à variation bornée sur Λ_f, telle qu'on ait

$$\|f(x) - \int e^{itx} d\mu(t)\| < \epsilon.$$

B. Sz.-Nagy (Szeged).

Rudin, Walter. Uniqueness theory for Laplace series. Trans. Amer. Math. Soc. **68**, 287–303 (1950).

The definition of Riemann summability for Laplace series $\sum_{n=1}^{\infty} Y_n(p)$ (Y_n is of degree n, p is on the unit sphere S) is based on the "Laplacian" $\psi f(p)$ where $f(p)$ is a function defined on S, $f(p)\epsilon L$. We form the mean $M(h)$ of $f(p)$ along a circle about p of spherical radius h and define $\psi f(p) = \lim_{h\to 0} 4h^{-2}(M(h) - f(p))$. More generally we define $\psi^* f(p)$ and $\psi_* f(p)$ in the same fashion replacing lim by lim sup and lim inf, respectively. We denote by Z a set of capacity zero on the sphere. The main result can be formulated then as follows. Suppose $-\sum Y_n(p)/n(n+1)$ is the Laplace series of a continuous function $f(p)$, $\psi^* f(p)$ and $\psi_* f(p)$ finite on $S-Z$. Further suppose the existence of a function $y(p)\epsilon L$ such that $y(p) \leqq \psi^* f(p)$ on S. Then the limit $\psi f(p)$ exists almost everywhere on S and $\sum_{n=1}^{\infty} Y_n(p)$ is the Laplace series of this limit. A similar result holds replacing the operator $\psi f(p)$ by the Poisson sum of $\sum_{n=1}^{\infty} Y_n(p)$.

G. Szegö (Stanford University, Calif.).

Men'šov, D. On partial sums of series of orthogonal functions. Učenye Zapiski Moskov. Gos. Univ. **135**, Matematika, Tom II, 3–9 (1948). (Russian)

Let $\varphi_1(x), \varphi_2(x), \cdots, \varphi_n(x), \cdots$ be an orthonormal system of functions in an interval (a, b), and let $p_1, p_2, \cdots, p_m, \cdots$ be an increasing sequence of positive integers such that $\sup (p_m - p_{m-1}) = +\infty$. The author shows that one can change the order of the functions within the system $\varphi_n(x)$ so that the resulting system $\{\varphi_{\nu_n}(x)\}$ has the following property: for any sequence $\{c_n\}$ with $\sum c_n^2 < \infty$ the sums $\sum_{n=1}^{p_m} c_n \varphi_{\nu_n}(x)$ tend almost everywhere to a finite limit. From this the following earlier result of the author is deduced: within any orthonormal system $\{\varphi_n(x)\}$ we may change the order so that $\sum c_n \varphi_{\nu_n}(x)$ is summable (C, ϵ), $\epsilon>0$, almost everywhere, for any c_n with $\sum c_n^2 < \infty$ [see Rec. Math. [Mat. Sbornik] N.S. **8(50)**, 121–136 (1940); these Rev. **2**, 281].

A. Zygmund (Chicago, Ill.).

Gelbaum, Bernard R. On the functions of Haar. Ann. of Math. (2) **51**, 26–36 (1950).

Using monotone sequences of partitions of the unit interval I into finitely many disjoint measurable sets, the author essentially characterizes the Haar functions by the following theorem. Let $F = \{f_n\}$ $(n \geqq 0)$ be a family of functions in $L(I)$ which satisfy (a) $f_0 \equiv 1$, (b) F is an orthonormal set, (c) the projection operators $P_n(f) = (\int_0^1 f(t) f_n(t) dt) f_n(x)$ all have norm 1, (d) the projection operators $S_n(f) = \sum_1^n P_k(f)$ all have norm 1, (e) the linear closure of F is $L(I)$. Then there exists a measure-preserving one-to-one point transformation $\varphi(x)$, defined almost everywhere and carrying I onto itself and such that $f_n(\varphi(x))$ are the standard Haar functions arranged in some order. In the author's words, the remaining sections of the paper treat the relationship between the Haar functions and spaces of vector-valued functions as well as some aspects of noncomplemented manifolds and the function families which cannot serve as their spanning vectors.

N. J. Fine (Philadelphia, Pa.).

Edwards, R. E. A property of a class of functions regular in the unit circle and a theorem on translations. J. London Math. Soc. 25, 33–39 (1950).

(1) Let C be the Banach space of the functions $\varphi(z)$ regular in $|z|<1$ and continuous in $|z|\leqq 1$, with $\|\varphi\|=\max|\varphi(z)|$ in $|z|\leqq 1$. Now let $\varphi(z)=\sum_0^\infty c_n z^n \epsilon C$ be fixed, and let E be a set contained in $|z|\leqq 1$. It is proved that the set $\{\varphi(\zeta z)\}$, $\zeta\epsilon E$, is complete (fundamental) in C, if (i) $c_n\neq 0$, $n=0, 1, \cdots$, and (ii) E is such that every $F(z)\epsilon C$ which vanishes on E vanishes identically. [Sufficient conditions for E different from those mentioned by the author are well-known [see, e.g., Blaschke, Ber. Verh. Sächs. Ges. Wiss. Leipzig. Math.-Phys. Kl. 67, 194–200 (1915)]. This answers a question of the author. Many results closely related to the author's are known, e.g., Boas, Ann. of Math. (2) 47, 21–32 (1946); these Rev. 7, 425; Ibragimov, Izvestiya Akad. Nauk SSSR. Ser. Mat. 13, 45–54 (1949); these Rev. 10, 604; Al'per, Doklady Akad. Nauk SSSR (N.S.) 66, 1029–1032 (1949); these Rev. 11, 24.] (2) Let C^* denote the Banach space (isometric with C: $\varphi(z)\leftrightarrow\varphi(e^{it})$) of the continuous periodic functions $f(t)\sim\sum_0^\infty c_n e^{int}$, with $\|f\|=\max|f(t)|$. If $f(t)\epsilon C^*$ is fixed, $c_n\neq 0$, $n=0, 1, \cdots$, and if T is a closed set of real numbers of positive Lebesgue measure, then the set $\{f(t-\tau)\}$, $\tau\epsilon T$, is complete in C^*. (3) Let C_p denote the Banach space of all continuous periodic functions $f(t)\sim\sum_{-\infty}^\infty c_n e^{int}$, with norm as before. Although the set of all translations of a fixed $f(t)\epsilon C_p$ for which $c_n\neq 0$, $n=0, \pm 1, \cdots$, is complete, there exists, to any proper closed subset T of $[0, 2\pi]$, a function $f(t)\epsilon C_p$ with $c_n\neq 0$, $n=0, \pm 1, \cdots$, such that $\{f(t-\tau)\}$, $\tau\epsilon T$, is not complete in C_p.

J. Korevaar (Lafayette, Ind.).

Polynomials, Polynomial Approximations

Erdös, P., and Turán, P. On the distribution of roots of polynomials. Ann. of Math. (2) 51, 105–119 (1950).

The authors prove the following theorem on equidistribution of the arguments $\varphi_1, \cdots, \varphi_n$ of the roots of a polynomial $f(z)=a_0+a_1z+\cdots+a_nz^n$. If $N(\alpha, \beta)$ denotes the number of φ's in the interval $\alpha\leqq\varphi\leqq\beta$ $(0\leqq\alpha<\beta\leqq 2\pi)$, then $|N(\alpha, \beta)-(2\pi)^{-1}n(\beta-\alpha)|<16(n\log P)^{\frac{1}{2}}$, where

$$P=|a_0a_n|^{-\frac{1}{2}}(|a_0|+\cdots+|a_n|).$$

The authors then show that two known theorems are consequences: (1) a theorem of E. Schmidt on the maximal number of real roots [S.-B. Preuss. Akad. Wiss. Phys.-Math. Kl. 1932, 321] and (2) a theorem of Szegö on the equidistribution of the roots of partial sums of a power series whose radius of convergence is 1 [S.-B. Berlin. Math. Ges. 21, 59–64 (1922)].

N. G. de Bruijn (Delft).

Specht, W. Abschätzungen der Wurzeln algebraischer Gleichungen. Math. Z. 52, 310–321 (1949).

Let the zeros $z_1, \cdots, z_n$ of the polynomial

$$f(z)=z^n+a_1z^{n-1}+\cdots+a_n$$

be labelled so that $|z_1|\geqq\cdots\geqq|z_n|$ and let the norm A of $f(z)$ be defined by the equation $A^2=1+|a_1|^2+\cdots+|a_n|^2$. In the first part of this paper the author establishes that $|z_1\cdots z_p|\leqq A$ and $|z_p|\leqq|A^{1/p}|$ for $p=1, 2, \cdots, n$. His proof is based upon the introduction of complex constants α_j for which

$$\sum_{j=1}^p\alpha_j+\sum_{k=1}^n a_k\sum_{j=1}^p\alpha_j z_j^{-k}=\sum_{j=1}^p\alpha_j z_j^{-n}f(z_j)=0$$

and for which therefore the Hermitian form

$$H(\alpha)=\sum_{j,k=1}^p[(A^2-1)(z_j\bar{z}_k-1)^{-1}-1]\alpha_j\bar{\alpha}_k\geqq 0.$$

The determinant of this form is

$$D=(A^2-1)^{p-1}\prod(z_j-z_k)\prod(z_j\bar{z}_k-1)^{-1}(A^2-|z_1z_2\cdots z_p|^2)\geqq 0.$$

The author shows also that, for a fixed A and variable n, the above limits on the zeros may be approached arbitrarily closely on making n sufficiently large. In the second part the author applies the results of the first part and mathematical induction to prove that, if $\alpha=\max|a_p^{1/p}|$ for $p=1, 2, \cdots, n$, then $|z_1\cdots z_p|\leqq p^{-p/2}(p+1)^{(p+1)/2}\alpha p=B$ and $|z_p|\leqq B^{1/p}$. These theorems reduce to well-known results when $p=1$.

M. Marden (Milwaukee, Wis.).

Meiman, N. N. Concerning a note of L. B. Geller. Uspehi Matem. Nauk (N.S.) 4, no. 6(34), 194–195 (1949). (Russian)

In the note of Geller [same vol., no. 2(30), 206–208 (1949); these Rev. 10, 702] it is stated that up to the present time necessary and sufficient conditions for all the roots of a polynomial to be real and negative have not been found. This statement is criticized by the author, who points out that necessary and sufficient conditions can be obtained (a) from the Sturm series for a polynomial and (b) from the theory of quadratic forms. The author further suggests that the important problem is to determine necessary and sufficient conditions that all the roots are real, since if all the roots are real, then a necessary and sufficient condition that they all are negative is that all the coefficients have the same sign.

A. W. Goodman (Lexington, Ky.).

Love, E. R. Addendum: On the zeros of partial sums of a certain power series. J. London Math. Soc. 25, 80 (1950).

Referring to his recent paper [same J. 24, 112–120 (1949); these Rev. 11, 30] the author acknowledges the prior and better result of Szegö [Jber. Deutsch. Math. Verein. 40, 163–166 (1931)] which states that $F_m(z)\neq 0$ for $|z|\leqq r=1$.

M. Marden (Milwaukee, Wis.).

Bottema, O. The mean-value theorem of the integral calculus for polynomials in two variables. Nieuw Arch. Wiskunde (2) 23, 108–110 (1950). (Dutch)

Let a_n be the largest zero of Legendre's polynomial $P_n(x)$. L. Tchakaloff [C. R. Acad. Sci. Paris 192, 32–35 (1931)] proved that if $f(x)$ is a real polynomial of degree $2n-2$ or $2n-1$, then there is a ξ satisfying $-a_n\leqq\xi\leqq a_n$ $(-a_n<\xi<a_n$ if $n\geqq 3)$ such that $f(\xi)=\frac{1}{2}\int_{-1}^1 f(x)dx$. The bounds for ξ are best possible. Now let $f(P)=f(x, y)$ be a real polynomial of total degree N; $4n-4\leqq N<4n$. Set $x^2+y^2=r^2$. The author proves that there is a point P, with r satisfying $\frac{1}{2}-\frac{1}{2}a_n\leqq r^2\leqq\frac{1}{2}+\frac{1}{2}a_n$, such that $f(P)=\pi^{-1}\int f(Q)dQ$, where the integral is extended over the unit-circle. The bounds for r^2 are best possible (for each N). [The author has "$<$" instead of "$\leqq$" in the inequalities for r^2, which is correct provided that $n\geqq 3$.] Proof: set $f(x, y)=\Phi(r)+\Psi(x, y)$, where $\Phi(r)$ is the average of $f(x, y)$ on the circle $x^2+y^2=r^2$. For each r there will be a point P such that $\Psi(P)=0$. Then $\Phi(r)$ is a polynomial in r^2 of degree less than $2n$. Setting $r^2=\frac{1}{2}+\frac{1}{2}t$, $\Phi(r)=\varphi(t)$, it follows from Tchakaloff's result that there is a point ξ, $-a_n\leqq\xi\leqq a_n$, such that

$$\int f(Q)dQ=2\pi\int_0^1\Phi(r)rdr=\tfrac{1}{2}\pi\int_{-1}^1\varphi(t)dt$$
$$=\pi\varphi(\xi)=\pi\Phi\{(\tfrac{1}{2}+\tfrac{1}{2}\xi)^{\frac{1}{2}}\}=\pi f(P).$$

[The corresponding problem in $k\geqq 3$ dimensions (for integrals over unit-spheres) can in the same way be reduced to the problem of finding best possible bounds for a point τ such that $\int_0^1 f(t)g(t)dt=f(\tau)\int_0^1 g(t)dt$, where $f(t)$ is a polynomial of degree not greater than $2n-1$ or not greater than $2n-2$, and $g(t)=t^{\frac{1}{2}k-1}$. A complete solution of the latter problem (with arbitrary $g(t)$) was given by L. Tchakaloff, C. R. Acad. Sci. Paris **192**, 330–333 (1931); compare also J. Favard, Bull. Soc. Math. France **59**, 229–255 (1931), who mentions the general 2-dimensional problem.]

J. Korevaar (Lafayette, Ind).

Berman, D. L. The divergence of S. N. Bernštein's interpolation process. Doklady Akad. Nauk SSSR (N.S.) **70**, 181–184 (1950). (Russian)

A classical result of the theory of interpolation asserts that there are continuous functions $f(x)$, $-1\leqq x\leqq 1$, such that their interpolating Lagrange polynomials corresponding to the Tchebysheff abscissas diverge at some points. S. Bernstein [Comm. Inst. Sci. Math. Méc. Univ. Kharkoff [Zapiski Inst. Mat. Mech.] (4) **5**, 49–57 (1932)] has given an interpolation process $A_n(f, x)$, corresponding to Tchebysheff abscissas and such that $A_n(x, f)$ converges uniformly to $f(x)$ for every continuous $f(x)$. His polynomial A_n is of degree higher than n, but the ratio of the degree of A_n to n can be as close to 1 as we please. The author now shows that the Bernstein procedure fails when applied to equidistant (Newton) abscissas. More precisely, the polynomials corresponding to the function $f(x)=x$ $(-1\leqq x\leqq 1)$ diverge for every $x\neq 0$. *A. Zygmund* (Chicago, Ill.).

Delcourte, M. Sur les suites de polynomes ${}_nP_c$ et ${}_nR_c$ de M. P. A. Pizá. Mathesis **58**, 309–325 (1950).

The polynomials in question were defined by P. A. Pizá by means of the following recursion formulas:

$$\begin{gathered}{}_nP_c=(cx+y)_{n-1}P_c+{}_{n-1}P_{c-1},\quad {}_0P_0=1,\quad {}_nP_c=0\ (c>n);\\ {}_nR_c=(cx+y)_{n-1}R_c+[(n-c+1)x-y]_{n-1}R_{c-1},\\ {}_0R_0=1,\quad {}_nR_c=0\ (c>n),\end{gathered}$$

and various properties thereof were stated by him without proof [same vol., 159–163 (1949); these Rev. **11**, 74]. The present author proves these properties, as well as some additional ones, by the use of generating functions and the methods of finite differences. He also gives an explicit expression for the polynomials. For $x=1$, $y=0$, ${}_nP_c$ is a "Stirling number of the second kind" and ${}_nR_c$ is a "Kummer number." The author thus obtains as a by-product numerous relations between these two sets of numbers.

H. W. Brinkmann (Swarthmore, Pa.).

Sansone, Giovanni. Su una disuguaglianza di P. Turán relativa ai polinomi di Legendre. Boll. Un. Mat. Ital. (3) **4**, 221–223 (1949).

Let $P_n(x)$ be the nth Legendre polynomial. The following inequality of Turán was discussed recently by the reviewer [Bull. Amer. Math. Soc. **54**, 401–405 (1948); these Rev. **9**, 429]:

$$\Delta_n(x)=(P_n(x))^2-P_{n-1}(x)P_{n+1}(x)\geqq 0,\qquad n\geqq 1,\ -1\leqq x\leqq 1.$$

Deriving a differential equation for $\Delta_n(x)$, the author proves the more informative estimate

$$(1-x^2)\Delta_n(x)>\tfrac{1}{2}\left(\int_x^1 P_n(x)dx\right)^2,\qquad n\geqq 1,\ -1<x<1.$$

It is shown also that $\Delta_n(x)+(P_n(x))^2/2n(n+1)$ is decreasing in (0, 1) from which fact upper and lower bounds for $\Delta_n(x)$ follow. *G. Szegö* (Stanford University, Calif.).

Gatteschi, Luigi. Una formula asintotica per l'approssimazione degli zeri dei polinomi di Legendre. Boll. Un. Mat. Ital. (3) **4**, 240–250 (1949).

The author obtains the following approximation [suggested by recent investigations of Tricomi, Ann. Mat. Pura Appl. (4) **26**, 283–300 (1947); these Rev. **10**, 700] of the rth root x_r of $P_n(x)$ (ordered in a decreasing manner):

$$x_r=\cos\frac{4r-1}{4n+2}\pi\left\{1-\frac{2n+7}{2(2n+1)(2n+3)(2n+5)}\right\}+\epsilon,$$

where $|\epsilon|<5.4n^{-4}$. Here $[\frac{1}{2}(n+2)]+1\leqq r\leqq[n/2]$. The proof is based on the asymptotic formula of Stieltjes (with 3 terms and remainder). The accuracy of this result is illustrated by the following consequence for the zero x_8 of $P_{18}(x)$: $0.61779<x_8<0.61797$. According to the Lowan-Davids-Levenson tables we have $x_8=0.61787$. *G. Szegö.*

Castoldi, Luigi. Sopra alcune proprietà dei polinomi di Legendre. Atti Accad. Ligure **5**, 99–110 (1949).

The author obtains various inequalities for the zeros ξ_i of Legendre's polynomial $P_n(x)$ and for the zeros η_i of $P_n'(x)$. (1) Let $\eta_{i-1}<\xi_i<\eta_i$. The area of $|P_n(x)|$ between η_{i-1} and ξ_i is the same as that between ξ_i and η_i. (2) The maxima of $|P_n(x)|$ decrease as x moves from 1 to 0. (3) Let $0<\eta_{i-1}<\xi_i<\eta_i$. Then $\xi_i-\eta_{i-1}>\eta_i-\xi_i$. (4) Let $0<\xi_i<\eta_i<\xi_{i+1}$, $P_n(\eta_i)>0$. Then η_i is nearer to ξ_{i+1} than to ξ_i. Also $P_n(\eta_i+h)<P_n(\eta_i-h)$, where $0<h\leqq\xi_{i+1}-\eta_i$. Various consequences concerning the location of zeros are pointed out. The reviewer refers to chapters VI and VII of his "Orthogonal Polynomials" [Amer. Math. Soc. Colloquium Publ., v. 23, New York, 1939; these Rev. **1**, 14] for certain results of the present paper. *G. Szegö.*

Pollaczek, Félix. Familles de polynomes orthogonaux. C. R. Acad. Sci. Paris **230**, 36–37 (1950).

In a previous note [same C. R. **228**, 1998–2000 (1949); these Rev. **11**, 104] the author dealt with the polynomials $P_n(x;\lambda,a,b)=P_n(x;\lambda)$ orthogonal with the weight

$$\begin{gathered}w(x;\lambda,a,b)=e^{(2\theta-\pi)\varphi(\theta)}|\Gamma(\lambda+i\varphi(\theta))|^2(1-x^2)^{\lambda-\frac{1}{2}};\\ x=\cos\theta,\quad 0\leqq\theta\leqq\pi,\quad \varphi(\theta)=\tfrac{1}{2}(a\cos\theta+b)/\sin\theta,\end{gathered}$$

$a>|b|$, $\lambda>-\frac{1}{2}$ (generalization of the ultraspherical polynomials). Let $\lambda=i\varphi(\theta_0)$, $\rho(a+be^{i\theta_0})=-ae^{i\theta_0}-b$. Then the polynomials $P(x;\lambda+1)-\rho P_{n-1}(x;\lambda+1)$ are orthogonal with the weight $(x-\cos\theta_0)w(x;\lambda,a,b)$. *G. Szegö.*

Geronimus, Ya. L. On certain polynomials of Steffensen. Doklady Akad. Nauk SSSR (N.S.) **69**, 721–724 (1949). (Russian)

Steffensen introduced three classes of polynomials $R_n^{[\lambda]}(x)$, $N_n^{[\lambda]}(x)$, $M_n^{[\lambda]}(x)$ by means of generating functions [Mat. Tidsskr. B. **1945**, 10–14; Acta Math. **78**, 291–314 (1946); these Rev. **7**, 157; **8**, 155]. It was pointed out in the cited reviews that $R_n^{[\lambda]}(x)$ is an Appell set and that the other two are sets of type one as defined by the reviewer [Duke Math. J. **5**, 590–622 (1939); these Rev. **1**, 15]. The author of the present paper comes essentially to the same conclusion, by pointing out that $R_n^{[\lambda]}(x)$ is a particular case of a class of Appell polynomials considered by him, and that the other two are Meixner polynomials [this is another name for polynomials of type one]. He shows how the properties of the Steffensen polynomials follow easily from the general properties of Geronimus and Meixner polynomials. [The reviewer's article referred to above contains a number of properties of type one polynomials.] *I. M. Sheffer.*

Mallet, M. Les suites isogènes. Houille Blanche 5, 7–14 (1950).

An isogenic sequence $\{f_n(x)\}$, $n=0, 1, \cdots$ is one for which $f_n'(x)=f_{n-1}(x)$. The only isogenic sequences treated here are polynomial sets with f_n of degree n, so they are Appell sets, although no mention is made of this name, nor is the well-known generating function (1) $A(t)e^{xt}=\sum_0^\infty f_n(x)t^n$ for Appell sets used (as it could be to advantage). Some general facts regarding Appell (i.e., isogenic polynomial) sets are found [mostly known, and that can be directly obtained from (1) above]. Then a detailed study is made of the two functional equations

$$(2)\qquad P_n(x+1)-P_n(x)=\Omega_{n-1}(x);\quad V_n(x+1)+V_n(x)=\Omega_n(x)$$

($n=0, 1, \cdots$). If $\{\Omega_n\}$ is an Appell set, each of equations (2) has an Appell set as a solution. Particular cases of $\{\Omega_n\}$ are considered, leading to solutions associated with the Bernoulli and Euler polynomials and numbers. The work closes with a set of problems based on this material.

I. M. Sheffer (State College, Pa.).

Special Functions

Sandham, H. F. A logarithmic transcendent. J. London Math. Soc. 24, 83–91 (1949).

The author discusses properties of the function

$$\psi(x)=\sum_{n=1}^{\infty} n^{-2}x^n \qquad (-1\leqq x\leqq 1)$$

in terms of the logarithmic integral

$$\mu(x)=\int_0^x \log^2(1-t)d\log t$$

and the related transcendent $M(x)=\mu(x)-\frac{1}{2}\log x\log^2(1-x)$. (It is to be understood that $\log w$ denotes the real part of the logarithm.) Some simple relations between μ functions depending on a single parameter x are derived, but the principal aim of the paper is to establish the existence of a relation between functions depending on n distinct parameters. The general result is embodied in the following theorem. If y is chosen small enough that the roots $x_1, \cdots, x_n$ of the equation $(1-\lambda_1 x)\cdots(1-\lambda_n x)=y$ are real when $\lambda_1, \cdots, \lambda_n$ are real, then

$$\tfrac{1}{2}\sum\left\{M(\alpha x)+M(\beta x)+M\left[\frac{\alpha-\beta}{\alpha(1-\beta x)}\right]-M\left[\frac{x(\alpha-\beta)}{1-\beta x}\right]\right\}$$
$$-M(1-y)=(n^2-1)M(1)-\tfrac{1}{2}n\sum M(1-\alpha/\beta),$$

where α and β take all values $\lambda_1, \cdots, \lambda_n$ and x takes all values $x_1, \cdots, x_n$. The number of functions involved (excluding $M(1)=2\zeta(3)$) is $n^2+1+\frac{1}{2}n(n-1)$. The formula for $n=2$ is written out in full.

M. C. Gray.

Lauwerier, H. A. A note on a logarithmic transcendant. Nieuw Arch. Wiskunde (2) 23, 163–169 (1950).

The author introduces four functions of which the first two are

$$\varphi(x)=\int_0^x t^{-1}[\log(1-t)]^2dt-\tfrac{1}{2}\log x[\log(1-x)]^2,\quad 0\leqq x\leqq 1,$$

$$\psi(x)=\int_0^x t^{-1}[\log(1+t)]^2dt-\tfrac{1}{2}\log x[\log(1+x)]^2,\qquad x\geqq 0.$$

Between these functions he establishes five functional relationships. A sample is $\psi(x)=\varphi(x/(1+x))$. He notes that his relationships can be verified easily by differentiation, but he prefers another, much longer, and more heuristic, deduction. Some consequences of the relations give information about special values of the functions in question. In particular

$$\varphi(1)=8\varphi(\tfrac{1}{2})=2\zeta(3),\quad \varphi(-\tfrac{1}{2}+\tfrac{1}{2}\sqrt{5})=\tfrac{4}{5}\zeta(3),$$
$$\tfrac{7}{8}\zeta(3)\leqq\varphi(\tfrac{2}{3})\leqq\tfrac{8}{9}\zeta(3).$$

He conjectures that the only values of x for which $\varphi(x)$ has an elementary value are $0, \frac{1}{2}, -\frac{1}{2}+\frac{1}{2}\sqrt{5}, 1$.

A. Erdélyi (Pasadena, Calif.).

Koschmieder, Lothar. Die Krümmung des Schaubildes der Jacobischen Etafunktion. Anz. Öster. Akad. Wiss. Wien. Math.-Nat. Kl. 85, 210–214 (1948).

Der Verf. beweist, dass das Schaubild der Funktion

$$\vartheta_1(v)=2\sum_{n=1}^{\infty}(-1)^{n-1}q^{\frac{1}{4}(2n-1)^2}\sin(2n-1)\pi v,\qquad v=\tfrac{1}{2}u/K,$$

für $0<q\leqq 0.06822$ (d.h., $0<k\leqq\sqrt{(2/3)}$) gegen die v-Achse beständig hohl ist, und keine Wendepunkte besitzt ausser an den Stellen $v=m$ ($m=0, \pm 1, \pm 2, \cdots$). Hauptzüge des Beweises (für $0<k\leqq 2^{-\frac{1}{2}}$): Aus

$$\frac{1}{2K}\frac{\vartheta_1'(v)}{\vartheta_1(v)}=Z(u)+\frac{\operatorname{cn}u\operatorname{dn}u}{\operatorname{sn}u}$$

folgt

$$\vartheta_1''(v)=4K^2\vartheta_1(v)[Z+cd/s+(c^2/s^2+E/K)^{\frac{1}{2}}]$$
$$\times[Z+cd/s-(c^2/s^2+E/K)^{\frac{1}{2}}];$$
$$\frac{\vartheta_1(v)}{\sin\pi v}=2q^{\frac{1}{4}}\prod_{n=1}^{\infty}(1-q^{2n})(1-2q^{2n}\cos 2\pi v+q^{4n})>0.$$

Setzt man

$$\psi(u)=Z(u)+\frac{\operatorname{cn}u\operatorname{dn}u}{\operatorname{sn}u}-\left(\frac{\operatorname{cn}^2u}{\operatorname{sn}^2u}+\frac{E}{K}\right)^{\frac{1}{2}};$$
$$\psi(0)=0,\quad \psi(K)=-(E/K)^{\frac{1}{2}}<0,$$

so ist

$$\psi'(u)=s^{-2}(c^2+Es^2/K)^{-\frac{1}{2}}[cd+(c^2+Es^2/K)^{\frac{1}{2}}]^{-1}$$
$$\times[c^2d^2-(c^2+Es^2/K)^2].$$

Wegen: $(1+k'^2)E-2k'^2K=k^2k'^2\int_0^K(\operatorname{sn}^2u/\operatorname{dn}^2u)du>0$ ist $E/K>2k'^2/(1+k'^2)>k'^2$, und

$$c^2d^2-(c^2-Es^2/K)^2<-s^2d^2(k'^4-k^2c^2)<0$$

für $k'^2\geqq k^2$ oder $k^2\leqq\frac{1}{2}$. Also $\psi(u)<0$ für

$$0<k\leqq 1/\sqrt{2}\rightarrow\theta''(v)/\sin\pi v<0,$$

$0<k\leqq 1/\sqrt{2}$. Unter bedeutend mehr Rechenarbeit wird dieser Schranke vorgetrieben bis $(2/3)^{\frac{1}{2}}$.

S. C. van Veen (Delft).

Humbert, Pierre. Image nouvelle pour la fonction de Bessel. C. R. Acad. Sci. Paris 230, 504–505 (1950).

With the notation

$$\nu(t, n)=\int_n^\infty\frac{t^x dx}{\Gamma(x+1)},$$

the author shows that the operational image of $I_t(x)$, where t is the variable and x is a parameter, is

$$p\sum_{m=0}^{\infty}\frac{x^m e^{mp}}{2^m m!}\nu(\tfrac{1}{2}xe^{-p}, m).$$

A. Erdélyi (Pasadena, Calif.).

Meijer, C. S. Neue Integraldarstellungen für Besselsche Funktionen. Compositio Math. 8, 49–60 (1950).

[This paper was submitted in 1939.] The author obtains integral representations of the form

$$\int_M I_{\alpha-\beta}(u^2)f(uz)u^{3-4\alpha-2\beta}du$$

for $K_\nu(z^2)$, $J_\nu(z^2)$, and $Y_\nu(z^2)$. Here M is a path of integration from $\infty e^{\frac{1}{2}\pi i}$ to $\infty e^{-\frac{1}{2}\pi i}$ entirely in the right half-plane. The functions $f(uz)$ in the three cases are defined and their properties are investigated. Particular values of α and β lead to a considerable number of specialized formulae (11 for K, 12 for J, and 7 for Y) in which f can be expressed in terms of Bessel functions. *A. Erdélyi* (Pasadena, Calif.).

Harries, W. Zwei Sätze über die Nullstellen der Bessel-Funktionen. Z. Angew. Math. Mech. 29, 381–382 (1949).

By consideration of the heat conduction equation the author deduces the formulae

$$\sum_{k=0}^{\infty}\alpha_k^{-2}=\{4(p+1)\}^{-1}, \quad \sum_{k=0}^{\infty}\alpha_k^{p-1}/J_{p-1}(\alpha_k)=\tfrac{1}{2},$$

where the α_k are the positive zeros of $J_p(z)$, $p=n/2-1$, and n is an integer. In a footnote he reproduces a remark by W. Magnus to the effect that the first of these two formulae is a simple consequence of the well-known factorisation of Bessel functions of the first kind. [Reviewer's remark: the second formula is a simple consequence of the so-called Kneser-Sommerfeld expansion; cf. H. Buchholz, same Z. 25/27, 245–252 (1947); these Rev. 9, 282.] *A. Erdélyi.*

Buchholz, Herbert. Besondere Reihenentwicklungen für eine häufig vorkommende zweireihige Determinante mit Zylinderfunktionen und ihre Nullstellen. Z. Angew. Math. Mech. 29, 356–367 (1949). (German. English, French and Russian summaries)

The author obtains various expansions for

$$X_\nu^{(p)}(z; R, r)=2\int_0^{\log(R/r)}\cosh(\nu\phi)(zw)^{-p}J_p(zw)d\phi,$$

where $w^2=R^2+r^2-2Rr\cos\phi$. First he expands in powers of z and shows that the coefficients can be expressed in terms of Legendre functions, then he uses a hypergeometric series expansion for $\cosh(\nu\phi)$ and obtains an expansion of X in terms of Bessel functions whose order is half of an odd integer, and finally he uses the addition theorem for $(zw)^{-p}J_p(zw)$ to obtain an expansion of X in a series of products of Bessel functions. These expansions, and a few further ones, are used to investigate the zeros (in z) of $J_\nu'(Rz)Y_\nu'(rz)-Y_\nu'(Rz)J_\nu'(rz)$ and the zeros (in ν) of $J_\nu(Rz)Y_\nu(rz)-Y_\nu(Rz)J_\nu(rz)$. In the course of the investigation the author discovers a zero of the first expression which he states has been overlooked by previous workers in the field. *A. Erdélyi* (Pasadena, Calif.).

Buchholz, Herbert. Uneigentliche Integrale mit parabolischen Funktionen über einen der beiden Parameter. Math. Z. 52, 355–383 (1949).

In this paper the author discusses certain infinite integrals which promise to be of importance in wave theory associated with a parabolic cylinder or a paraboloid of revolution. The investigation is mathematical, and the setup is more general than applications to wave theory demand, although the general form of the integrals has been retained. The first integral studied by the author is of the form

$$\int\Gamma(a+s)\Gamma(a-s)u^{2s}M_{k+s,m}(-ix)M_{k'+s,m'}(iy)ds,$$

where integration is extended over one or the other of the so-called Bromwich contours. Other integrals are obtained by replacing one or both M-functions in a suitable manner by W-functions, or else by using a product of four confluent hypergeometric functions. There are also other useful generalisations which do not lend themselves to a brief description. In spite of some overlapping with a recent paper by the author [Z. Physik 124, 196–218 (1948); these Rev. 10, 453] and with an earlier paper by the reviewer [Proc. Roy. Soc. Edinburgh. Sect. A. 61, 61–70 (1941); these Rev. 3, 116], the most general results of the present paper appear to be new. *A. Erdélyi* (Pasadena, Calif.).

Toscano, Letterio. Sviluppi in serie della funzione ipergeometrica di Kummer. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 6, 590–597 (1949).

Of the three principal results of this paper the first, equation (4), gives the sum of the series

$$\sum_{m=0}^{\infty}\frac{\Gamma(\beta+m)}{\Gamma(\gamma+m)}x^m L_m^{(\alpha)}(y);$$

the second, equation (5), is a generalisation of the generating function of Laguerre polynomials which is already known [A. Erdélyi, Math. Z. 42, 641–670 (1937), equation (5, 7)]; and the third, equation (17), is a multiplication theorem of confluent hypergeometric functions which is also known [Erdélyi, ibid., 125–143 (1936), equation (5, 1)]. Numerous particular cases of these expansions are written out. *A. Erdélyi* (Pasadena, Calif.).

Merbt. Wave propagation from moving sources: Methods for numerical calculation of the functions $h_{mn}(\alpha, \sigma)$. Untersuchung zur Arbeit von H. G. Küssner: "Lösungen der klassischen Wellengleichung für bewegte Quellen." Methode zur numerischen Berechnung der Funktionen $h_{mn}(\alpha, \sigma)$. Air Materiel Command, Wright-Patterson Air Force Base, Dayton, Ohio, F-TS-5863-RE (ATI-32413), i+13 pp.+14 pp. (1948).

[This is a reproduction and a translation of a report ZWB/AVA/RE/44/J/31, ZWB 10514, from the Aerodynamische Versuchsanstalt, Göttingen.] The function

$$h_{m,n}(\alpha, \sigma)=\frac{1}{2\pi}\int_0^{2\pi}\frac{\exp\{imn[-\chi-\alpha(1-2\sigma\cos\chi)^{\frac{1}{2}}]\}}{(1-2\sigma\cos\chi)^{\frac{1}{2}}}d\chi$$

occurs in an investigation by Kuessner [Z. Angew. Math. Mech. 24, 243–250 (1944); these Rev. 10, 80] of sound propagation from a source moving in a circle. In the present report first an asymptotic expansion

$$h_{mn}(\alpha, \sigma)=i^{mn}\sum_{\nu=1}^{\infty}b_\nu J_{mn}(\nu mn\alpha\sigma),$$

then a convergent expansion in powers of σ are obtained. According to the author, the convergent expansion is more suitable for purposes of numerical computation, but it is too involved to be reproduced here. The first seven b_ν and fourteen coefficients for the convergent expansion are given explicitly. *A. Erdélyi* (Pasadena, Calif.).

Sips, Robert. Représentation asymptotique des fonctions de Mathieu et des fonctions d'onde sphéroidales. Trans. Amer. Math. Soc. **66**, 93–134 (1949).

Der Verf. beschäftigt sich mit asymptotischen Entwicklungen der Mathieu'schen und der Spheroidalfunktionen für grosse Werte des Parameters, welche im Gegensatz zu früheren Darstellungen die Funktionen für alle reellen Werte der unabhängigen Variablen darstellen. Bei den Mathieu'schen Funktionen geht Verf. von der algebraischen Form der Differentialgleichung aus und erhält für die asymptotische Lösung eine lineare inhomogene Differentialgleichung zweiter Ordnung. Die Lösungen werden in der üblichen Weise normiert. Für die Lösungen selber und für die Eigenwertparameter erhält er asymptotische Ausdrücke, deren erste Glieder angewendet werden. Die Genauigkeit dieser Darstellung wird numerisch mit der exakten Lösung verglichen. Bei den Spheroidalfunktionen wendet Verf. das analoge Verfahren, sowohl beim gestreckten als auch beim abgeplatteten Ellipsoid an. Zur besseren Anwendung der Lösungen gibt Verf. eine Tabelle der Koeffizienten seiner asymptotischen Reihen. Auch hier vergleicht er numerisch die asymptotischen Werte der Eigenwertparameter mit den exakten Werten und er führt diesen Vergleich auch für die Funktionswerte durch. *M. Strutt* (Zürich).

Sips, Robert. Solution générale de l'équation de Mathieu. Bull. Soc. Roy. Sci. Liége **18**, 220–235 (1949).

Sips, Robert. Solution générale de l'équation de Mathieu. II. Bull. Soc. Roy. Sci. Liége **18**, 289–299 (1949).

Verf. befasst sich sowohl mit den Lösungen der Mathieu'schen als auch mit jenen der assoziierten Mathieu'schen Funktionen. In beiden Fällen betrachtet er die beiden unabhängigen Lösungen der betreffenden Differentialgleichungen. Er schickt einige Integralrelationen allgemeiner Art voraus und wendet diese dann auf die Lösung der zweidimensionalen Wellengleichung in elliptischen Koordinaten an. Hierdurch erhält er allgemeine Formeln für die oben genannten Lösungen mit Hilfe bestimmter Integraldarstellungen. Die Integranden enthalten Bessel'sche und Hankel'sche Funktionen. *M. Strutt* (Zürich).

Campbell, Robert. Sur une expression remarquable des solutions de période $2k\pi$ de l'équation de Mathieu associée. Bull. Soc. Math. France **77**, 1–9 (1949).

Verf. wendet die von Poole auf die Mathieu'sche Differentialgleichung angewendete Methode an auf den Fall der assoziierten Mathieu'schen Differentialgleichung. Diese Methode beruht auf einem Theorem von Fuchs über die Entwicklung der Lösungen in der Umgebung der singulären Punkte. Hierdurch gelangt er zu Darstellungen der Lösungen mit verschiedener Periodizität. Schliesslich geht er auf die numerische Anwendung seiner Darstellung ein.

M. Strutt (Zürich).

Harmonic Functions, Potential Theory

Nehari, Zeev. Note on positive harmonic functions. J. London Math. Soc. **25**, 19–26 (1950).

(1) Let $f(z)=\sum_{n=-\infty}^{\infty}a_nz^n$, $a_0=1$, have positive real part in $\rho<z<1$. Then $|a_n|\leqq 2/(1-\rho^n)$, $|a_{-n}|\leqq 2\rho^n/(1-\rho^n)$, $n>0$. (2) The general harmonic function in $\rho<r<1$ is $\gamma\log r+\sum_{n=-\infty}^{\infty}r^n(a_n\cos n\theta+b_n\sin n\theta)$; consider such functions which are positive and have $a_0=1$. Then

$$(a_n^2+b_n^2)^{\frac{1}{2}}\leqq 2/(1-\rho^n)+2\gamma\rho^n(1-\rho^{2n})^{-1}\log\rho,$$

$n>0$, with a corresponding inequality for $n<0$. (3) Let $u(r,\theta,\phi)$ be harmonic and positive for $\rho<r<1$,

$$u=1+\sum_{n=1}^{\infty}\{S_n(\theta,\phi)r^n+S_{-n}(\theta,\phi)r^{-n}\}.$$

Then $|S_n(\theta,\phi)|\leqq(2n+1)/(1-\rho^n)$, $n>0$, with a corresponding inequality for $n<0$. All these inequalities are best possible. *R. P. Boas, Jr.* (Providence, R. I.).

Gabriel, R. M. An inequality concerning three-dimensional subharmonic functions. J. London Math. Soc. **24**, 309–312 (1949).

The author extends to the 3-dimensional case a result given for 2 dimensions by Fejér and F. Riesz [Math. Z. **11**, 305–314 (1921)] and Frazer [same J. **6**, 113–117 (1931)]. Theorem: if $U(P)$ is subharmonic and nonnegative for P inside and on a sphere K of which D is a diametral plane, then for $\lambda\geqq 2$, $\iint_D U^\lambda(P)dS\leqq\iint_K U^\lambda(P)dS$. Of course it suffices to prove the theorem when U is harmonic (of any sign) and $\lambda=2$. The proof uses a precise majorization involving Hilbert's inequality. *L. Schwartz* (Nancy).

Gabriel, R. M. Some inequalities concerning integrals of two-dimensional and three-dimensional subharmonic functions. J. London Math. Soc. **24**, 313–316 (1949).

The author proved in a previous article [same vol., 154–156 (1949); these Rev. **11**, 108] that, if U is nonnegative and subharmonic in the volume V interior to the closed convex surface Γ, one has $|U^\lambda(P)|\leqq 2\Omega^{-1}\int_\Gamma|U^\lambda(Q)|d\Omega_P(Q)$, where $d\Omega_P(Q)$ is the elementary solid angle subtended on Γ at P; Ω is the area of the unit sphere. Let now C be a surface in V, $\chi(Q)$ the absolute total solid angle $\int_C|d\Omega_Q(P)|$ subtended on C at Q. By an estimation of the ratio between $d\Omega_P(Q)$ on Γ and $d\Omega_P(P)$ on C, and application of classical Hölder inequality, the author obtains, for $\lambda>2$: $\int_C|U^\lambda(P)|dS\leqq A(\lambda)\int_\Gamma|U^\lambda(Q)|\chi(Q)\Omega^{-1}dS$. An estimation of the constant $A(\lambda)$ is given. This formula is interesting especially if C approaches the boundary Γ. If C is a diametral plane of a sphere Γ, one obtains a theorem, previously given by the author [see the preceding review] with a better constant A and for $\lambda\geqq 2$. An application is given to the analytic functions of a complex variable, generalizing an inequality of Carlson [Ark. Mat. Astr. Fys. **29B**, no. 11 (1943); these Rev. **6**, 205]. *L. Schwartz* (Nancy).

Pólya, Georges. Sur la symétrisation circulaire. C. R. Acad. Sci. Paris **230**, 25–27 (1950).

L'auteur définit la symétrisation circulaire (analogue à celle de Steiner) comme une transformation d'un domaine S et un domaine S^*, relativement à un demi-plan P limité par la droite d. À l'ensemble e des points de S situés sur une circonférence γ d'axe d, correspond un arc de γ, de même longueur et de milieu dans P. La réunion de ces arcs constitue S^*. Cette opération conserve le volume, minore la surface-frontière et fournit sur l'intégrale de Dirichlet à 2 variables une inégalité dont on indique diverses applications. Par exemple soient un domaine plan δ limité par des courbes assez régulières (Γ externe, γ_i internes), u harmonique dans δ s'annulant sur les γ_i et prenant la valeur 1 sur Γ; l'auteur introduit $(4\pi)^{-1}\int_\Gamma(du/dn)ds$ sous le nom de capacité de δ [ce n'est pas la capacité classique, mais par exemple ce que Brelot a appelé contenance de Γ relativement au domaine infini limité par les γ_i, voir J. Math. Pures Appl. (9) **24**, 1–32 (1945), p. 25; ces Rev. **7**, 521]. Si Γ est une circonférence fixe, les γ_i des arcs d'une même circonférence concentrique fixe et de longueur totale l, cette

capacité spéciale est minima quand il n'y a qu'un seul arc γ de longueur l. L'article comporte quelques démonstrations et des énoncés. *M. Brelot* (Grenoble).

Ursell, F. Surface waves on deep water in the presence of a submerged circular cylinder. II. Proc. Cambridge Philos. Soc. 46, 153–158 (1950).

In order to fill in a gap in part I [same vol., 141–152 (1950); these Rev. 11, 480], the author proves the following theorem concerning two-dimensional potential functions. Let φ satisfy the Laplace equation in the part of the half-plane $y\geqq 0$ outside the circle $|z-if|=a$, where $a<f$, and also the boundary conditions $\partial\varphi/\partial n=0$ on $|z-if|=a$, $K\varphi+\partial\varphi/\partial y=0$ on $y=0$, where $K>0$ and $(\partial\varphi(\partial x)^2+(\partial\varphi/\partial y)^2\to 0$ as $|z|\to\infty$ $(y\geqq 0)$. Then $\varphi\equiv 0$. *J. V. Wehausen* (Providence, R. I.).

Cattaneo, Carlo. Sul calcolo di alcuni potenziali e sul loro intervento nella risoluzione di particolari problemi armonici. Atti Sem. Mat. Fis. Univ. Modena 3, 29–45 (1949).

Let D be the disk $x^2+y^2<a^2$, $z=0$, on which there is a mass distribution of superficial density $\mu=(1-\rho^2/a^2)^{\frac{1}{2}}$, where $\rho^2=x^2+y^2$. The Newtonian potential on D of this distribution is of the form $a+b\rho^2$. If the density is μ^{-1}, the potential on D is constant (μ^{-1} is the conductor density). By means of a second-order recurrence relation, the author is able to evaluate explicitly the potential when the density is of the form μ^n, n being any odd positive integer; it is a polynomial of degree $(n+1)/2$ in ρ^2. It is shown that, by linear combinations of potentials of these distributions and their partial derivatives, it is possible to represent the solution of the mixed boundary value problem for the upper half space, in which the normal derivative is zero on that portion of the (x, y)-plane outside D, while the values to be assumed on D are those of a prescribed polynomial in x and y. The paper concludes with an application to the study of the small oscillations of a disk in a moving fluid. *J. W. Green.*

Pignedoli, Antonio. Su alcune equazioni differenziali del primo ordine in cui intervengono le funzioni armoniche, e corrispondenti applicazioni meccaniche. Atti Sem. Mat. Fis. Univ. Modena 3, 3–9 (1949).

L'auteur tire quelques exercices scolaires de remarques telles que la suivante: si $U(x, y)$ est harmonique, de gradient G, U_x-iU_y est une fonction holomorphe $f(z)$ $(z=x+iy)$ et U_ydx-U_xdy vaut $|G|^2\Re dz/f(z)$. Application en particulier au mouvement du point (x, y) soumis à la force $|G|^{-2}U_x$, $-|G|^{-2}U_y$. *M. Brelot* (Grenoble).

Pignedoli, Antonio. Sui potenziali logaritmici. Atti Sem. Mat. Fis. Univ. Modena 3, 10–13 (1949).

L'auteur exprime la densité u d'une couche de masses sur une circonférence Γ au moyen du potentiel logarithmique $V(\theta)$ de ces masses sur Γ selon

$$u(\theta')-\frac{1}{4\pi^2R\log R}\int_0^{2\pi}Vd\theta-\frac{1}{4\pi^2R}\int_0^{2\pi}\frac{Vd\theta}{\sin^2\frac{1}{2}(\theta-\theta')}.$$

Le calcul est basé sur une formule de Baggio donnant la dérivée du potentiel selon le rayon vecteur et suppose l'existence de $V''(\theta)$. *M. Brelot* (Grenoble).

Ascoli, Guido. L'isotropia analitica e le sue applicazioni. Univ. e Politecnico Torino. Rend. Sem. Mat. 8, 109–122 (1949).

Partant du caractère linéaire et invariant par rotation de l'harmonicité, l'auteur introduit les notions de système et opérateur linéaires isotropes. Le système désigne, dans une couronne plane de centre O, une famille de fonctions continues contenant leurs combinaisons linéaires, les fonctions déduites de l'une f par rotation autour de O, ainsi que $\int f(\rho, \theta+\alpha)\varphi(\alpha)d\alpha$ (ρ, θ coordonnées polaires, $\varphi(\alpha)$ continue de période 2π). Un opérateur linéaire isotrope sur les fonctions du système précédent les change en fonctions continues, est linéaire et se permute avec une rotation et même avec l'opérateur définie par l'intégrale ci-dessus. Ces considérations sont appliquées à la recherche de "solutions simples" et de développements de Fourier, au moins formels, en retrouvant des résultats connus. L'auteur termine par une extension à l'espace où l'angle polaire est remplacé par un point de la "variété des rotations." *M. Brelot.*

Willmore, T. J. Mean value theorems in harmonic Riemannian spaces. J. London Math. Soc. 25, 54–57 (1950).

A Riemannian space is said to be centrally harmonic at a point P if Δ_2S is a function of s only, where s is the geodesic distance from P, and Δ_2 is the second Beltrami operator. It is shown that if a space is centrally harmonic at P, the mean value on a geodesic sphere with centre at P of any harmonic function in the space is equal to the value at the centre. Conversely, it is shown that if every harmonic function has the property that its mean value over every geodesic sphere with centre at P is equal to its value at P, then the space is centrally harmonic at P. Thus a new definition of a centrally harmonic space is obtained. It is also shown that if a space is centrally harmonic at every point, a necessary and sufficient condition that a function be harmonic is that its mean value on every geodesic sphere is equal to its value at the centre. *W. V. D. Hodge.*

Tihonov, A. N. On the uniqueness of the solution of the problem of electric prospecting. Doklady Akad. Nauk SSSR (N.S.) 69, 797–800 (1949). (Russian)

A point-source at O (origin, $x=y=z=0$) creates in a half-space $z\geqq 0$ of conductibility σ a cylindrically symmetric field $u(r, z)$, $r=(x^2+y^2)^{\frac{1}{2}}$. The author considers the case when σ is a function of z only, $\sigma=\sigma(z)$, so that $u(r, z)$ satisfies the equation

$$r^{-1}\frac{\partial}{\partial u}\left(r\frac{\partial u}{\partial z}\right)+\sigma^{-1}\frac{\partial}{\partial z}\left(\sigma\frac{\partial u}{\partial z}\right)=0$$

with the usual boundary conditions. Using the substitutions $\zeta=\int_0^z\sigma(t)dt$ and $Z(z)=-\sigma(z)(d/dz)\int_0^\infty u(r, z)J_0(\lambda r)rdr$, where J_0 denotes the Bessel function of the first kind and zero order, the author shows that $Z(z)$ is a solution of the equation $\sigma^2(z)Z''(z)=\lambda^2Z(z)$ which enables him to prove the uniqueness of the solution: to different laws of conductibility $\sigma_1(z)$, $\sigma_2(z)$ correspond on the boundary plane $z=0$ different potential distributions $u_1(r, 0)$ and $u_2(r, 0)$. *E. Kogbetliantz* (New York, N. Y.).

Radenković, Dragoš. A solution of the problem of steady state stresses for a rectangular domain. Glas Srpske Akad. Nauka 195, 89–104 (1949). (Serbian)

It is shown how the Dirichlet value problem for biharmonic functions with given boundary values for the function and its normal derivative can be solved if the boundary values are given by Fourier series and the domain is mapped conformally onto the unit circle. Numerical applications are given for the case of rectangular domains. *W. Feller* (Ithaca, N. Y.).

Differential Equations

Bol, G. Invarianten linearer differentialgleichungen. Abh. Math. Sem. Univ. Hamburg 16, 1–28 (1949).

The chief results may be summarized as follows. (I) The basic elements are the "half invariants of weight m" which from the geometrical point of view may be looked upon as covariant tensors of valence m in one dimension. A connection τ may be found which enables the construction of a covariant derivative $g=g'-m\tau g$ of any half invariant of weight m. All parameters which make $\tau=0$ are related by a projective transformation and the covariant derivative is equal to the ordinary derivative with respect to any such projective parameter.

(II) A given equation

$$(1)\qquad z^{(n+1)}=\sum_{j=1}^{n+1} b_j(t)z^{(n+1-j)}(t)$$

may be simplified to

$$(2)\qquad x^{(n+1)}=\sum_{j=2}^{n+1} a_jx^{(n+1-j)},$$

where $x=\rho z$ and ρ is a conveniently chosen factor. The form of (2) is preserved by a parameter transformation, provided x is a half invariant of weight $-n/2$.

(III) Let α_k, $\alpha_{k\mu}$ be conveniently chosen numerical constants and e_ρ a half invariant of weight ρ; then in the system associated with (2):

$$(3)\qquad P_k'=P_{k+1}+\alpha_k aP_{k-1}+\sum_{\mu=0}^{k}\alpha_{k\mu}e_{\mu+1}P_{k-\mu}$$

(with $a=a_2/N$, where $N\neq0$ is a conveniently chosen number) the e's may be determined in a unique way so that $P_{n+1}=0$ and (3) may be looked upon as Frenet formulas for the curve given by (2). (The e's do not depend on the solution of a differential equation for τ which leads to $a_2^*=0$ by $t^*\to t$.)

(IV) Similar results hold for the hyperplane coordinates of the osculating hyperplanes of the curve. [Reviewer's remark. Basically the same system of Frenet formulas in a projective curved $(n-1)$-dimensional space was obtained by the reviewer [Abh. Sem. Vektor- und Tensoranalysis [Trudy Sem. Vektor. Tenzor. Analizu] 2/3, 119–144 (1935)].

V. Hlavatý (Bloomington, Ind.).

Tricomi, Francesco. Un nuovo metodo di studio delle equazioni differenziali lineari. Univ. e Politecnico Torino. Rend. Sem. Mat. 8, 7–19 (1949).

This is an expository article devoted to a new method, which is credited to Fubini, for solving linear ordinary differential equations. Taking a homogeneous equation of the second order, for example, the equation is supposed to be written in the form

$$(1)\qquad y''+p_1(x)y'+p_2(x)y=A(x)y''+B(x)y'+C(x)y,$$

where

$$(2)\qquad y''+p_1(x)y'+p_2(x)y=0$$

is an equation with a known general solution. If $y_1=F_1(x)$, $y_2=F_2(x)$ are two linearly independent solutions of (2), the solution of (1) is written in the form

$$y=C_1(x)F_1(x)+C_2(x)F_2(x),$$

and the method of variation of constants is applied formally, the right-hand member of (1) being treated as a known function. This results in the functions $C_1(x)$, $C_2(x)$ being determined in terms of the solution of a certain integral equation of the Volterra type. The latter equation can be solved by the usual method of successive approximations. Examples are given to show that this method for solving differential equations can be used very satisfactorily to obtain general results concerning the asymptotic properties of solutions, and concerning eigenvalues and eigenfunctions of boundary value problems. Other examples illustrate the use of the method to obtain properties of specific functions, such as Bessel functions, hypergeometric functions, etc.

L. A. MacColl (New York, N. Y.).

Wintner, Aurel. On linear asymptotic equilibria. Amer. J. Math. 71, 853–858 (1949).

The system of linear differential equations $x_i'=\sum_{k=1}^{n}a_{ik}x_k$, $i=1,\cdots,n$, is said to be of type (*) if as $t\to\infty$, $x_i(t)\to C_i$, $i=1,\cdots,n$, where C_i are constants and moreover there is a solution corresponding to any set of C_i. A sufficient condition for type (*) is: (a) $\int^\infty a_{ik}(t)dt$ exists as $\lim\int^T$, $T\to\infty$; (b) $\int^\infty|\int_s^\infty a_{ik}(t)|ds<\infty$; and (c) $\limsup|a_{ik}(t)|<\infty$ as $t\to\infty$. Another sufficient condition is: (a) as above, (d) $\int^\infty|\int_s^\infty a_{ik}(t)dt|^pds<\infty$; and (e) $\int^\infty|a_{ik}(t)|^qdt<\infty$ where $p>1$ and $q=p/(p-1)$. [The limiting case $p=\infty$ is known, as the author indicates.]

N. Levinson (Cambridge, Mass.).

Ricci, Lelia. Sulle vibrazioni quasiarmoniche di un sistema dissipativo. Univ. e Politecnico Torino. Rend. Sem. Mat. 8, 191–208 (1949).

The equation $y''+2cy'+w^2(t)y=0$ is studied, where $c>0$ is a constant and $w(t)=m$, $0\leq t<\frac{1}{2}T$, $w(t)=M$, $\frac{1}{2}T\leq t<T$; $0<m<M$ and $w(t)$ has period T.

N. Levinson.

Putnam, C. R. An oscillation criterion involving a minimum principle. Duke Math. J. 16, 633–636 (1949).

Let $q(x)$ be real and continuous on $0\leqq x<\infty$. Let $y(x)$ be real and continuous and $y'(x)$ piecewise continuous for $a\leqq x<\infty$, where $a\geqq0$, and let $y(a)=0$. Let $\int_a^\infty y^2dx=1$ and let $\int_a^\infty(y'^2+|q|y^2)dx<\infty$. Let $\mu(a)$ denote the greatest lower bound of $\int_a^\infty(y'^2+qy^2)dx$ for the class of $y(x)$ described above. Clearly $\mu(a)$ is nondecreasing. Then if and only if $\mu(a)<0$ for all $a\geqq0$, the differential equation $y''-qy=0$ is oscillatory on $0\leqq x<\infty$, that is, every solution has an infinite number of zeros.

N. Levinson (Cambridge, Mass.).

Nardini, Renato. Sul comportamento asintotico degli integrali di un'equazione differenziale della dinamica. I. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 7, 47–52 (1949).

Nardini, Renato. Sul comportamento asintotico degli integrali di un'equazione differenziale della dinamica. II. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 7, 52–61 (1949).

The author studies the asymptotic behavior of the solutions of the equation $u''+p(t)u'+q(t)u=0$. Under certain hypotheses concerning p and q, he shows that all solutions tend to zero as $t\to\infty$. The main idea is to convert the original equation into a form where the coefficient of u' is missing, or alternatively to compare its solutions with those of a new equation where $p(t)$ is taken to be constant.

R. Bellman (Stanford University, Calif.).

Greco, Donato. Su un problema ai limiti per un'equazione differenziale lineare ordinaria del secondo ordine. Giorn. Mat. Battaglini (4) 2(78), 216–237 (1949).

The equation $(p(x)y')'+(\lambda r(x)-q(x))y=0$ is considered, where p, q and r are continuous and positive over $a\leqq x\leqq b$.

The boundary conditions are

(C) $K_1y=(\alpha_1\lambda+\alpha_1')y(a,\lambda)+(\alpha_2\lambda+\alpha_2')y'(a,\lambda)=0$

and a similar condition $K_2y=0$ at b where the α's are replaced by β's. Here the α's and β's are constants and it is assumed that $\alpha_1\alpha_2'-\alpha_1'\alpha_2>0$ while $\beta_1\beta_2'-\beta_1'\beta_2<0$. The case (A) where $K_1y=y(a,\lambda)$ is also considered as is the case (B) where $K_2y=y(b,\lambda)$. Theorems are given concerning the location of the eigenvalues in the cases (A) and (B) as compared with the location of eigenvalues in the case $y(a,\lambda)=y(b,\lambda)=0$. The case (C) is then treated by locating its eigenvalues in comparison with (B). *N. Levinson.*

Hartman, Philip. A characterization of the spectra of one-dimensional wave equations. Amer. J. Math. 71, 915–920 (1949).

The equation (*) $(px')'+(q+\lambda)x=0$ is considered, where $p(t)>0$ and $q(t)$ are both real and continuous, $0\leqq t<\infty$. The limit point case is assumed and a boundary condition at $t=0$, $x(0)\cos\alpha+x'(0)\sin\alpha=0$. The spectrum is denoted by S. Let $x(t,\lambda)$ be a real solution and let $N(t,\lambda)$ be the number of zeros of $x(t,\lambda)$ on $0\leqq t<T$. Let $\lambda'<\lambda''$ and let n denote lim inf $(N(T,\lambda'')-N(T,\lambda'))$ as $T\to\infty$. Then there are exactly n points of S satisfying $\lambda'<\lambda<\lambda''$ or $\lambda'\leqq\lambda<\lambda''$ according as (*) is oscillatory or nonoscillatory for $\lambda=\lambda''$. Moreover $n=\infty$ if and only if there are an infinite number of points of S in $\lambda'<\lambda<\lambda''$. In the proof it is shown that if $\mu_0(T),\mu_1(T),\cdots$ denote the eigenvalues of the finite Sturm-Liouville problem on $0\leqq t\leqq T$ with $x(T)=0$ then λ belongs to S if and only if $d(T)\to 0$ as $T\to\infty$ where $d(T)=\min|\lambda-\mu_j(T)|$ for $j=0,1,2,\cdots$. *N. Levinson.*

Langer, Rudolph E. The asymptotic solutions of ordinary linear differential equations of the second order, with special reference to a turning point. Trans. Amer. Math. Soc. 67, 461–490 (1949).

For large $|\lambda|$ asymptotic expansions in the interval $a\leqq x\leqq b$ are obtained for the solutions of the equation

(1) $d^2u/dx^2+[\lambda^2q_0(x)+\lambda q_1(x)+R(x,\lambda)]u=0,$

if $R(x,\lambda)=\sum_{\nu=0}^{\infty}v_\nu(x)/\lambda^\nu$ and q_0, q_1 and v_ν are indefinitely differentiable. In particular the case where $q_0(x)$ has a simple zero is considered. Previously only the principal terms of the asymptotic expansions were known in the special case $q_1(x)\equiv 0$. By means of a series of successive transformations a related equation is derived with known solutions, the coefficient in the standard form (after removing the term with the first derivative) agreeing with the coefficient in (1) in all terms of the expansion for $R(x,\lambda)$ of order lower than λ^{-m} (m is an arbitrary positive integer).

The first approximating equation is obtained by the usual substitution $\Phi(x)=q_0^{\frac{1}{2}}(x)$, $\xi=\lambda\int_0^x\Phi(t)dt$ (the zero of q_0 is taken as origin), $\Psi(x)=[\int_0^x\Phi(t)dt]^{1/6}/[\Phi(x)]^{\frac{1}{2}}$. Using the function $V(\xi)$, which satisfies the equation $d^2V/d\xi^2+\frac{1}{3}\xi^{-1}dV/d\xi+V=0$ and hence is of the form $\xi^{\frac{1}{3}}H_{\frac{1}{3}}(\xi)$, where H stands for any Bessel function of order $\frac{1}{3}$, a function $v(x,\lambda)=\Psi(x)V(\xi)$ is formed, which satisfies

(2) $d^2v/dx^2+[\lambda^2q_0(x)-\Psi''/\Psi]v=0.$

If $q_1(x)\neq 0$ a second approximating equation is formed by forming the differential equation satisfied by the functions

$$\zeta_j(x,\lambda)=\left\{\mu_0(x)v_j(x,\lambda)+\frac{\mu_1(x)}{\lambda}v_j'(x,\lambda)\right\},\quad j=1,2,$$

where v_j $(j=1,2)$ is a set of solutions of (2) and the functions $\mu_0(x)$ and $\mu_1(x)$ are determined by the requirement that the resulting equation for ζ_j, in the standard form without first derivative, agrees with (1) in the terms with order λ^2 and λ. Denoting the corresponding solutions of this standard form by $z_j(x,\lambda)$, the author finds higher approximations by repeating this process.

Introduce $\eta_j(x,\lambda)=A(x,\lambda)z_j(x,\lambda)+\lambda^{-2}B(x,\lambda)z_j'(x,\lambda)$, where $A(x,\lambda)$ and $B(x,\lambda)$ are polynomials in λ^{-1} with unknown coefficients. A set of recursion formulae for these coefficients is obtained by requiring that the differential equation satisfied by η_j $(j=1,2)$, in the standard form without first derivative (in which case the $\eta_j(x,\lambda)$ become the functions $y_j(x,\lambda)$), agrees with (1) in all terms of order lower than λ^{-m}. The known expressions $y_j(x,\lambda)$ then are the asymptotic expansions of the solutions of (1). For the remainder terms in these asymptotic expansions an integral equation of the Volterra type is derived, which permits a rigourous estimate. The last part of the paper contains asymptotic forms of the remainder term for different ranges of arg λ and different choices of the Hankel functions which describe the behaviour of the solutions in intervals close to the turning point and intervals at a finite distance from this point. *J. G. van der Corput* (Amsterdam).

Kodaira, Kunihiko. The eigenvalue problem for ordinary differential equations of the second order and Heisenberg's theory of S-matrices. Amer. J. Math. 71, 921–945 (1949).

E. C. Titchmarsh [Eigenfunction Expansions Associated with Second-Order Differential Equations, Oxford, 1946, especially chapter III; these Rev. 8, 458] obtained a "concrete" spectral theorem concerning the spectral decomposition connected with an ordinary second order differential operator. The proof given there is based on classical analysis. In the present paper a proof is given employing the abstract spectral theory of von Neumann, Riesz and Stone as well as earlier results of H. Weyl. [The author states that he obtained his results independently of Titchmarsh and other recent literature which has not been available to him and that he revised the paper after his attention had been drawn to these publications.] In addition the general theory is applied to special questions (criteria for discreteness of the spectrum) and special equations (Hill's and Schroedinger's equation).

E. H. Rothe (Ann Arbor, Mich.).

★**Nielsen, Kaj Leo. General Boundary Value Problems for Linear Differential Equations.** Abstract of a Thesis, University of Illinois, 1940. ii+12 pp.

This study continues investigations made by Trjitzinsky on differential systems of the nth order with linear boundary conditions at two or more points. In particular, it studies cases where the differential equation (linear) is nonhomogeneous and the coefficients are asymptotically representable as power series in negative powers of the parameter with coefficients that are functions of the independent variable. Under suitable hypotheses on the systems with boundary conditions at two points, it is shown that the nonhomogeneous system is compatible for all sufficiently large values of the parameter. A similar result is established for the system when the boundary conditions apply to a denumerable set of points of the interval. The principal study of Trjitzinsky upon which the dissertation is based appeared in Acta Math. 67, 1–50 (1936). *W. M. Whyburn.*

***Hamming, Richard Wesley. Some Problems in the Boundary Value Theory of Linear Differential Equations.** Abstract of a Thesis, University of Illinois, 1942. ii+7 pp.

This dissertation studies linear differential systems with two point boundary conditions of the type discussed by Trjitzinsky [Acta Math. **67**, 1–50 (1936)]. In particular, it obtains expressions for the Green's function and asymptotic expressions for the characteristic solutions. Methods developed by Tamarkin [Math. Z. **27**, 1–54 (1927)] are extended to study the distribution of the characteristic numbers and to establish uniform convergence of the series used over the regions described. *W. M. Whyburn* (Chapel Hill, N. C.).

El'šin, M. I. On linear systems with established spherical motion. Učenye Zapiski Moskov. Gos. Univ. **135**, Matematika, Tom II, 173–187 (1948). (Russian)

The transformation $x=\rho\cos\phi$, $y=\rho\sin\phi$, $\rho>0$, connects the linear system $\dot{x}=a_{11}(t)x+a_{12}(t)y$, $\dot{y}=a_{21}(t)x+a_{22}(t)y$ with the nonlinear system $\dot{\phi}=\frac{1}{2}(a_{21}-a_{12})+R(t)\cos\frac{1}{2}(2\phi+\psi)$, $\dot{\rho}/\rho=\frac{1}{2}(a_{11}+a_{22})+R(t)\sin\frac{1}{2}(2\phi+\psi)$, where

$$R=\{(a_{11}-a_{22})^2+(a_{12}+a_{21})^2\}^{\frac{1}{2}},$$
$$\cos\psi=(a_{21}+a_{12})/R,\quad \sin\psi=(a_{11}-a_{22})/R.$$

By means of this transformation the author discusses the question of the boundedness of the solutions of the original system. Thus, for example, if $\int^\infty R(t)dt<\infty$, and $\int^\infty(a_{11}+a_{22})dt$ is convergent, then all solutions are bounded. Many similar results are given. *R. Bellman.*

Malkin, I. G. On stability under constantly acting disturbances. Amer. Math. Soc. Translation no. **8**, 8 pp. (1950).

Translated from Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **8**, 241–245 (1944); these Rev. **7**, 298.

Levenson, Morris E. Harmonic and subharmonic response for the Duffing equation $x+\alpha x+\beta x^3=F\cos\omega t$ $(\alpha>0)$. J. Appl. Phys. **20**, 1045–1051 (1949).

Two types of harmonic and subharmonic solutions are determined and studied by the perturbation method for small β. A comparison of the perturbation and Rauscher methods is made. *N. Levinson* (Cambridge, Mass.).

Got, Théophile. Détermination des solutions périodiques stables de certaines équations différentielles quasi harmoniques. C. R. Acad. Sci. Paris **230**, 612–614 (1950).

Poincaré showed that the system $\dot{x}=y$, $\dot{y}=-x+\mu P(x,y)$ has a limit-cycle, for all small $\mu>0$; here $P(x,y)$ is a polynomial, $P(-x,-y)=-P(x,y)$. The author introduces polar coordinates p,θ and obtains the expression $p=p_0+\sum_{n=1}^{\infty}\mu^n p_n(\theta)$ for a limit cycle; the functions $p_n(\theta)$ have period π and are calculated by recursion. Similar expressions are obtained for the period and amplitude of a limit-cycle. The explicit results of these calculations are presented for the equation of van der Pol, in which $P(x,y)=(1-x^2)y$. *J. G. Wendel* (New Haven, Conn.).

Cafiero, Federico. A proposito dell'equazione di Clairaut modificata. Boll. Un. Mat. Ital. (3) **4**, 257–260 (1949).

An equation of the type $y'=f(x,y)$, where f is continuous in the rectangle R: $a\leqq x\leqq b$, $c\leqq y\leqq d$, is called a modified equation of Clairaut if through every point of the rectangle there passes a line with the property that the part of the line contained in R is a solution. The author demonstrates a comparison theorem which can be used to establish a uniqueness theorem previously proved by Ważewski and Szarski [Ann. Soc. Polon. Math. **20** (1947), 157–160 (1948); these Rev. **10**, 121]. *R. Bellman.*

Stampacchia, Guido. Un'osservazione su un problema ai limiti per l'equazione: $y^{(n)}=\lambda f(x,y,y',\cdots,y^{(n-1)})$. Boll. Un. Mat. Ital. (3) **4**, 235–239 (1949).

The problem of determining a value λ such that a solution of $y^{(n)}=\lambda f(x,y,y',\cdots,y^{(n-1)})$ passes through n given points has been considered recently in many papers. The author makes the observation that this equation may be regarded as a first integral of the $(n+1)$th order equation

$$y^{(n+1)}=y^{(n)}\left\{\frac{\partial\log f}{\partial x}+\frac{\partial\log f}{\partial y}y'+\cdots+\frac{\partial\log f}{\partial y^{(n-1)}}y^{(n)}\right\},$$

and from this fact, under certain assumptions, is able to conclude the existence of the desired constant λ. *R. Bellman* (Stanford University, Calif.).

Backes, F. Sur un problème relatif aux formes de Pfaff. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **35**, 938–944 (1949).

Let $\omega=P_1dx_1+\cdots+P_ndx_n$ be a Pfaffian form in an n-dimensional space X_n and let ω' be its exterior derivative. Let f be a function defined on X_n. The author proves that ω is a total differential on the hypersurfaces $f=$constant if, and only if, f is a solution of the complete system $[\omega', df]=0$. This system admits nontrivial solutions if, and only if, $[\omega',\omega']=0$ everywhere in X_n and in that case the number of functionally independent solutions is two. [Remark by the reviewer: an extension of the author's results can be found in E. Goursat, Leçons sur le problème de Pfaff, Hermann, Paris, 1922, chapter IV, pp. 160, 161, 166, 167.] *W. van der Kulk* (Providence, R. I.).

Zervos, P. Sur l'intégration symbolique. Prakt. Akad. Athēnōn **15**, 448–450 (1940). (French. Greek summary)

Eine symbolische Differentialform Ω vom Grad $p-1$ nennt Verf. ein symbolisches Integral der symbolischen Differentialform ω p-ten Grades, wenn $\Omega^-=\omega$. (Bei Pfaffschen Formen bedeutet Ω^- die bilineare Kovariante im allgemeinen Falle deren von E. Goursat eingeführte Verallgemeinerung.) Der Begriff des symbolischen Integrals wird nunmehr zunächst auf Pfaffsche Systeme übertragen und anschliessend die symbolische Form Ω in $p+1$ Variablen vom Grade p behandelt, insbesondere ihre Reduktion auf eine "abgeleitete Form" durch Abänderung eines ihrer Koeffizienten mit Fallunterscheidungen je nachdem p gerade oder ungerade ist. Dabei gelangt Verf. auch zu symbolischen Integralen zweiter und höherer Ordnung ($\Omega^{=}=\Omega\Omega'=\omega$, $\Omega^{=}=\frac{1}{2}(\Omega^-)^2=\omega,\cdots$). *M. Pinl* (Dacca).

***Picard, Émile. Leçons sur quelques types simples d'équations aux dérivées partielles avec des applications à la physique mathématique.** Gauthier-Villars, Paris, 1950. iii+214 pp. 700 francs.

"Nouveau tirage" of a work first published in 1927.

Levinson, Norman. The first boundary value problem for $\epsilon\Delta u+A(x,y)u_x+B(x,y)u_y+C(x,y)u=D(x,y)$ for small ϵ. Ann. of Math. (2) **51**, 428–445 (1950).

Under certain mild conditions (see below) the solution in a domain R of the problem stated in the title is shown to be of the form $u(x,y,\epsilon)=U(x,y)+z(x,y,\epsilon)+O(\epsilon^{\frac{1}{2}})$, where $U(x,y)$ and $z(x,y,\epsilon)$ are defined as follows. Consider a curvilinear quadrilateral in R bounded by two characteristics of $Au_x+Bu_y+Cu=D$ and two arcs S_1, S_2 of

the boundary of R. It is assumed that S_1 and S_2 are nowhere tangent to the vector field with components A and B. Let S_1 be that arc on which the vectors (A, B) point into the exterior of R. If $\epsilon>0$, then $U(x, y)$ is that solution of $Au_x+Bu_y+Cu=D$ which takes on the given boundary values on S_1. The "boundary layer" term z is of the form $\exp[-g(x, y)/\epsilon]\cdot h(x, y)$ near S_2 and uniformly $O(e^{-\delta/\epsilon})$, $\delta>0$, in the remaining part of the quadrilateral. Here $g(x, y)$ is zero on S_2 and positive inside the quadrilateral. The functions g and h are solutions of certain first order partial differential equations.

In addition to certain smoothness requirements, which (as the author states) could be considerably weakened, the following condition is imposed. There exists a function $\Gamma(x, y)$ of class C'' on R and on its boundary such that there $A\Gamma_x+B\Gamma_y>0$. It is shown that this condition is always satisfied if A and B can be extended into a simply connected open domain R_0 containing R and its boundary in such a way that A and B are of class C' in R_0, and that $A^2+B^2\neq 0$ in R_0. The author sketches a generalization of his result for the case that A, B, C, D depend on ϵ. He also shows that his method yields, with only obvious modifications, a full asymptotic expansion for $u(x, y, \epsilon)$. The main tools of the proof are the maximum principle for elliptic differential equations, the theory of the characteristics of first order partial differential equations, and a method for proving the smallness of the remainder by means of an integral inequality. *W. Wasow* (Los Angeles, Calif.).

Viguier, Gabriel. Enchainement et quantification: L'équation de Schrödinger pour un oscillateur harmonique linéaire. Revue Sci. 86, 519–522 (1948).

Verfasser behandelt die Schrödinger-Gleichung

$$\Delta\psi-(8\pi^2m/h^2)V\psi-(4\pi im/h)(\partial\psi/\partial t)=0$$

für den harmonischen linearen Oszillator. Durch geeignete Transformation sowohl der abhängigen wie auch der unabhängigen Veränderlichen entsteht zunächst die Differentialgleichung $d^2a/dq^2+(\lambda-q^2)a=0$ und für $a(q)e^{q^2/2}=H_k(q)$ die Differentialgleichung der Hermiteschen Polynome vom Grade k. Für $H_k\tau=\partial H_k/\partial q$ verwandelt sich die Differentialgleichung der Hermiteschen Polynome schliesslich in eine Riccatische Differentialgleichung $d\tau/dq+\tau^2-2q\tau+\lambda-1=0$. Auf diese Weise gelingt es Verfasser Eigenschaften der Hermiteschen Polynome als geometrische Eigenschaften einer Riccatischen Differentialgleichung zu deuten. *M. Pinl.*

Agostinelli, Cataldo. Sulla determinazione degli autovalori nel problema delle vibrazioni di una membrana con contorno epicicloidale fisso. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 7 (1949), 316–320 (1950).

The solution of the partial differential equation of the vibrations of a membrane, $u_{xx}+u_{yy}+\lambda^2u=0$, is written down as an infinite series, with unknown coefficients, of products of Bessel functions and trigonometric functions of polar coordinates. New coordinates ρ, φ are introduced by the relation $x+iy=na\rho e^{i\varphi}+b\rho^ne^{in\varphi}$, n an integer, $0<a<b$. The inequalities $0\leqq\rho<1$, $0\leqq\varphi\leqq 2\pi$ describe the interior of an epicycloid; the boundary of the epicycloid is $\rho=1$, and here the boundary condition is $u=0$. Instead of enforcing the full boundary condition, the author demands only

$$\int_0^{2\pi} u \begin{matrix}\cos\\ \sin\end{matrix}\, \nu\varphi d\varphi=0$$

for $\rho=1$ and $\nu=0, 1, \cdots, m$. It turns out that only $2m+1$ coefficients enter in the $2m+1$ equations so obtained, and a determinantal equation is found for the approximate characteristic values λ. Another possible approach is indicated in the paper. If the expansion

$$u=\tfrac{1}{2}u_0(\rho)+\sum_{s=1}^{\infty}\{u_s(\rho)\cos s\varphi+v_s(\rho)\sin s\varphi\}$$

is substituted into the partial differential equation, one obtains an infinite system of ordinary linear differential equations for the u_s and v_s. The solutions of this system are called functions of the epicycloidal cylinder. *A. Erdélyi* (Pasadena, Calif.).

Brodin, Jean. Expression générale du principe de Huyghens pour les propagations amorties d'ondes longitudinales. C. R. Acad. Sci. Paris 229, 989–991 (1949).

Brodin, Jean. Expression générale du principe de Huyghens pour les ondes électromagnétiques en milieu imparfaitement transparent. C. R. Acad. Sci. Paris 229, 1064–1066 (1949).

Brodin, Jean. Application du principe de Huyghens au dioptre: expression des ondes réfléchie et réfractée. C. R. Acad. Sci. Paris 230, 67–69 (1950).

By the problem of Huyghens, the author means the determination of a distribution of sources on a closed surface S which shall produce the same wave motion outside S as a given distribution of sources inside S. The first note is concerned with damped harmonic waves specified by a scalar solution of $\Delta U=(a+ik)^2U$ which satisfies the conditions $RU\rightarrow 0$, R grad $U\rightarrow 0$ as $R\rightarrow\infty$. It is shown that there is only one solution of this equation regular outside S which is such that U or its normal derivative takes given values on S, and that this solution can be expressed as the effect either of a simple layer or of a double layer of sources on S, or more generally as the effect of a simple and a double layer, one of which is arbitrary. Kirchhoff's formula is the only one which gives a null effect inside S. The second note deals with the corresponding problem for damped harmonic electromagnetic waves, expressed in terms of the Hertzian vector. The results are applied in the third note to the problem of reflexion and refraction by a body of arbitrary form. *E. T. Copson* (Dundee).

Tihonov, A. On boundary conditions containing derivatives of order higher than the order of the equation. Mat. Sbornik N.S. 26(68), 35–56 (1950). (Russian)

The heat equation $\partial^2u/\partial x^2=\partial u/\partial t$ with conditions $\sum_{k=0}^{m}\alpha_k\partial^ku(0, t)/\partial x^k=f(t)$, $u(x, 0)=0$, is solved, and the solution examined especially for its stability properties. Avoiding all mention of the Laplace transformation, the author uses an awkward sequence of devices equivalent to it. For the characteristic equation in the form $\sum_{k=0}^{m}\alpha_kq^k=0$, the solution is stable if the roots q_i satisfy the condition $-3\pi/4<\arg q_i<3\pi/4$, and if $\lim_{t\rightarrow\infty}f(t)$ is finite. *R. E. Gaskell* (Ames, Iowa).

Lidyaev, S. F. On the representation of a solution of the equation of heat conduction in the form of a Poisson integral. Učenye Zapiski Moskov. Gos. Univ. 135, Matematika, Tom II, 86–109 (1948). (Russian)

Concerning the Poisson integral,

$$p(x, t)=\tfrac{1}{2}\pi^{-\frac{1}{2}}\int_{-\infty}^{\infty}t^{-\frac{1}{2}}e^{(x-\xi)^2/(4t)}\phi(\xi)d\xi,$$

the author first proves that if $p(x, t)$ converges for x_0, t_0,

(A) then for some positive numbers, $c, k, e^{-cx^2}|p(x, t)| < kt^{-\frac{1}{2}}$ for $0 \leqq t \leqq t_0$; (B) the integral $\int_{-\infty}^{\infty} e^{-cx^2} p(x, t)dx$ converges uniformly in t. Then he shows that if $u(x, t)$ is a positive parabolic function (solution of $\partial^2 u/\partial x^2 = \partial u/\partial t$) such that $e^{-cx^2}|u(x, t)| < kt^{-\frac{1}{2}}$ and for which $\int_{-\infty}^{\infty} e^{-cx^2} u(x, t)dx$ converges uniformly in t, then $u(x, t)$ can be written as a Poisson-Stieltjes integral

$$u(x, t) = \tfrac{1}{2}\pi^{-\frac{1}{2}} \int_{-\infty}^{\infty} t^{-\frac{1}{2}} e^{-(x-\xi)^2/(4t)} d\phi(e),$$

where the set function $\phi(e)$ is of bounded variation. Finally, if $u(x, t)$ is a positive parabolic function such that $\max_{0 \leqq t \leqq t_0} |u(x, t)| e^{-cx^2} < kt^{-\frac{1}{2}}$ and $\int_{-\infty}^{\infty} e^{-cx^2} u(x, t)dx$ converges uniformly in t, and further if $F(e, \xi) = \int_e u(x, \xi)dx$ has the property of uniform absolute continuity, then $u(x, t)$ is expressible in the form of a Poisson-Lebesgue integral for all t in the interval $0 < t < t_0 - \xi_0$. The proofs of these theorems and lemmas leading up to them are given in considerable detail. *R. E. Gaskell* (Ames, Iowa).

Landau, H. G. Heat conduction in a melting solid. Quart. Appl. Math. 8, 81–94 (1950).

Heat at the rate $H(t)$ per unit area is supplied at the moving face $x = s(t)$ of an infinite plane-parallel slab at initial temperature $T_0(x)$, the right face $x = a$ being insulated. The solid melts at temperature T_m, and, in contrast to other similar problems, the melted material is removed. After a discussion of the general problem, it is specialized by setting $H(t) = H_0$, $T_0(x) = T_0$, both constant, and letting $a \to \infty$. The specialized problem is simplified by elimination of the moving boundary through use of the variable $x - s(t)$, which represents the distance from the face of the melting solid. The new statement of the problem is especially convenient because it involves only the single parameter $m = \pi^{\frac{1}{2}} c(T_m - T_0)/2L$, where L is the heat of fusion, c the specific heat. The steady-state solution (case of uniform progress of the melting face) is given, as is the solution for $m = 0$. Through numerical integration, $s(t)$ is found for $m = 0.2, 1, 2, 5, 10$ and ∞. *R. E. Gaskell* (Ames, Iowa).

Garavaldi, Orestina. Su di un problema di propagazione termica, trattato col metodo degli operatori funzionali. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 6, 461–466 (1949).

The problem referred to in the title, and an analogous problem in electrical theory, lead to the solution

$$q(t) = \frac{\alpha\gamma}{\alpha\gamma + \coth \gamma s} Q(t),$$

where $Q(t)$ is a given function of time, α, s are constants and γ^2 is a constant multiple of d/dt. The author first obtains the solution for the case when $Q(t)$ is the integral of the unit function. The solution of the general case follows by Duhamel's theorem. The whole process runs parallel to the standard treatment of such problems [cf. for instance Carslaw and Jaeger, Conduction of Heat in Solids, Oxford, 1947, chapters III, XI, in particular, §§ 41, 117; these Rev. 9, 188] except that instead of the theory of the Laplace transformation, Giorgi's theory is used to justify the formal operations. *A. Erdélyi* (Pasadena, Calif.).

Courant, Richard, and Lax, Peter. On nonlinear partial differential equations with two independent variables. Comm. Pure Appl. Math. 2, 255–273 (1949).

Les auteurs considèrent ici un système quasi-linéaire de n équations à n fonctions inconnues, hyperboliques, dont les coefficients ont des dérivées premières et secondes continues; en se donnant les valeurs des n fonctions sur une portion de l'axe des x, ils sont conduits à un système d'équations intégrales, non linéaires, qu'ils résolvent par une méthode d'approximations successives; pour la clarté de l'exposé, ils se bornent d'abord au cas semi-linéaire, où les caractéristiques sont connues d'avance, et où les inconnues à introduire s'imposent immédiatement; le cas général est nettement plus difficile: chaque stade d'approximation implique un nouveau champ de caractéristiques; d'autre part, de nouvelles inconnues, surabondantes, sont introduites, et c'est après coup que l'on démontre que certaines d'entre elles sont identiquement nulles (raisonnement dont la principe général est susceptible d'applications variées). Malgré le caractère satisfaisant des résultats antérieurs de Schauder et de Friedrichs, le présent article a son abilité, non seulement à cause de la simplicité de la méthode, fondée sur les propriétés des caractéristiques, mais aussi parce que cette méthode s'applique à des problemes "mixtes," et parce qu'elle permet, comme le montrera un article ultérieur, d'obtenir assez simplement une solution numérique. *M. Janet* (Paris).

Lahaye, Edmond. Le problème de Cauchy et la résolution de certaines catégories d'équations linéaires du second ordre et d'ordres supérieurs à multiplicités caractéristiques décomposables. Acad. Roy. Belgique. Cl. Sci. Mém. Coll. in 8°. (2) 21, no. 8, 80 pp. (1949).

Let A_i^j, E be given functions of $x_1, \cdots, x_n$, and denote by Δ_i the operator $\Delta_i \equiv A_0^i + A_1^i \partial/\partial x_1 + \cdots + A_n^i \partial/\partial x_n$. By solving successively the set of linear equations $\Delta_1 z_1 = E$, $\Delta_2 z_2 = z_1, \cdots, \Delta_p u = z_{p-1}$, the author arrives at solutions of the pth order equation (1) $\Delta_1 \Delta_2 \cdots \Delta_p u = E$. A surface $\phi = 0$, with $\Delta_i - A_0^i \phi = 0$ holding for some i, is called a characteristic surface for (1). The Cauchy problem for (1) consists in finding a solution u which on the surface $\Gamma \equiv x_n - \Gamma_n(x_1, \cdots, x_{n-1}) = 0$ satisfies the conditions

$$(2) \qquad u = K_0, \quad \partial u/\partial x_n = K_1, \cdots, \partial^{p-1} u/\partial x_n^{p-1} = K_{p-1},$$

where the K_i are given functions of $x_1, \cdots, x_{n-1}$. The solution of (1) mentioned above can be used to solve this Cauchy problem if $\Gamma = 0$ is not a characteristic surface. In case $\Gamma = 0$ is a characteristic surface, the last condition in (2) is replaced by $u = L(x_1, \cdots, x_{n-2}, x_n)$ on the surface $\Gamma^1 \equiv x_{n-1} - \Gamma_{n-1}(x_1, \cdots, x_{n-2}, x_n) = 0$. Under certain restrictions on the surface $\Gamma^1 = 0$ this problem is shown to have a solution. A third type of problem is also treated. Here the last two conditions in (2) are replaced by giving the values of u on $\Gamma^1 = 0$ and the values of u on a third surface $\Gamma^2 = 0$. In this last problem $\Gamma = 0$, $\Gamma^1 = 0$ are characteristic surfaces of (1), and while $\Gamma^2 = 0$ satisfies certain restrictions it may or may not be a characteristic surface. In the last part of the paper the three problems noted above are solved for the equation (3) $\Delta_1 \Delta_2 \cdots \Delta_p u = E + H$, where H is a form linear in u and its derivatives of order less than p. The method of successive approximations is used to obtain the results in this part of the paper. *F. G. Dressel* (Durham, N. C.).

Functional Analysis, Ergodic Theory

Dixmier, J. Les fonctionnelles linéaires sur l'ensemble des opérateurs bornés d'un espace de Hilbert. Ann. of Math. (2) 51, 387–408 (1950).

This paper contains detailed proofs of results announced earlier [C. R. Acad. Sci. Paris 227, 948–950 (1948); these

Rev. 10, 307]. One of these results asserts that if $\mathfrak{B}'$ is the conjugate of the Banach space $\mathfrak{B}$ of all bounded linear operators on a Hilbert space then $\mathfrak{B}'$ is the direct sum of two specifically defined subspaces $\mathfrak{s}'$ and $\mathfrak{s}_\perp$ in such a way that the norm of each element is the sum of the norms of its components. Let K be the convex set of all "non-negative" elements in the unit sphere of $\mathfrak{B}'$. An appendix added in proof contains a demonstration of the fact that a point of K is an extreme point if and only if it is either an extreme point of $K\cap\mathfrak{s}'$ or an extreme point of $K\cap\mathfrak{s}_\perp$. The author acknowledges that a number of results are included in earlier work of Schatten and of Schatten and von Neumann [Ann. of Math. (2) 47, 73–84, 608–630 (1946); these Rev. 7, 455; 8, 31] which did not come to his attention until his paper was in the hands of the printers.
G. W. Mackey (Princeton, N. J.).

Hartman, Philip, and Wintner, Aurel. Separation theorems for bounded Hermitian forms. Amer. J. Math. 71, 865–878 (1949).

The authors give proofs of the following facts. Let H be a bounded Hermitian form in a Hilbert space, M a subspace, H_M the restriction of H to M. If $s_1<s_2$ are in the spectrum of H or H_M, respectively, there exists $s_1\leqq s\leqq s_2$ in the spectrum of H_M or H, respectively, provided M is of codimension one. Let $s_1(M)\leqq\cdots\leqq s_m(M)$ be the proper values of H_M (with repetitions), where now $m=\dim M<\infty$. Put $s_i'=\inf s_i(M)$ with respect to a collection C of M's such that, if M_1, $M_2\epsilon C$ then $M_1, M_2\subset M$ for some $M\epsilon C$, and the union of all M's in C is dense in the space. Let s' be the least element in the essential spectrum of H. The following are the only possibilities: either the spectrum of H has no element less than s', and then $s_i'=s'$ $(1\leqq i)$, or the spectrum of H has only a finite number of elements $s_1\leqq\cdots\leqq s_r<s'$ (with repetitions), and then $s_i'=s_i$ $(1\leqq i\leqq r)$ and $s_i'=s'$ $(r<i)$, or the spectrum of H has a sequence $s_1\leqq\cdots\leqq s_i\leqq\cdots<s'$ (with repetitions), and then $s_i'=s_i$ $(1\leqq i)$. *L. Nachbin.*

Štraus, A. V. On a class of regular operator-functions. Doklady Akad. Nauk SSSR (N.S.) 70, 577–580 (1950). (Russian)

Let H_1, H_2, H_1', H_2' be unitary spaces with $\dim H_1=\dim H_2'$, $\dim H_2=\dim H_1'$, and V an isometric transformation of $H_1\oplus H_2$ to $H_1'\oplus H_2'$ such that $V\phi_k\epsilon H_k'$ for $\phi_k\epsilon H_k$ implies $\phi_k=0$. Writing $\phi=\phi_1+\phi_2$, $\phi_i\epsilon H_i$, we have $V\phi_k=V_{1k}\phi_k+V_{2k}\phi_k$, where V_{ik} is a transformation from H_k to H_i'. Let $F(z)$ be a transformation from H_1 to H_1' such that: (1) $F(z)$ is a regular function of the complex parameter z for $|z|<1$, (2) $\|F(z)\|<1$, (3) $F(0)=V_{11}$. Then the transformation (4) $G(z)=V_{22}^*\overline{V_{12}^*F(z)[E-V_{11}^*F(z)]^{-1}V_{21}^*}$ is well-defined from H_2' to H_2, is a regular function of z for $|z|<1$, and (5) $\|G(z)\|\leqq|z|$. Conversely, if $G(z)$ has these last properties, $F(z)$ defined by (6) $F(z)=V_{11}+V_{12}G(z)[E-V_{22}G(z)]^{-1}V_{21}$ satisfies (1)–(4). Further, from a result of Neumark [Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] 4, 53–104 (1940); these Rev. 2, 104] it follows that if $H_1, H_1', F(z)$ with properties (1), (2) are given, then H_2, H_2', V, G can be defined so that (3), (4), (5), (6) hold. Conversely, if F can be written in the form (6), then (1) and (2) hold. *B. Crabtree* (Cambridge, Mass.).

Stampacchia, Guido. Le trasformazioni funzionali che presentano il fenomeno di Peano. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 7, 80–84 (1949).

Generalizing the situation with respect to the Peano existence theorem for a system of linear differential equations the author says that a transformation $y=T(x)$ on a linear normed complete space Σ to a space Σ' of the same kind presents the "phenomenon of Peano" if for every y the corresponding points x are either unique or constitute a continuum. The author proves that if the continuous transformation T is on a bounded connected closed domain d, if every sequence of Σ which is transformed into a convergent sequence in Σ' is compact, and if T can be approximated uniformly on d by a sequence of one-to-one continuous transformations $T_n(x)$ on d then $T(x)$ exhibits the phenomenon of Peano at every point y in the interior of $T(d)$ for which there exists a neighborhood belonging to all the sets $T_n(d)$. The theorem is applied to systems of differential equations and it is pointed out that it also applies to a general functional equation of Volterra type due to Tonelli [Bull. Calcutta Math. Soc. 20, 31–48 (1930)], viz., $\psi(y)=\varphi(x)+A(x,\varphi(y))$ where A depends on the values of $\varphi(y)$ between 0 and x. *T. H. Hildebrandt.*

Lorentz, G. G. Some new functional spaces. Ann. of Math. (2) 51, 37–55 (1950).

Two function spaces $\Lambda(\alpha)$ and $M(\alpha)$, $0\leqq\alpha\leqq1$, are considered. These consist of functions integrable over $0\leqq x\leqq1$ in which the norms are defined in the following ways. In $\Lambda(\alpha)$, $\|f\|_{\Lambda(\alpha)}=\alpha\int_0^1x^\alpha f^*(x)dx$, where $f^*(x)$ is a nondecreasing rearrangement of $|f(x)|$. In $M(\alpha)$,

$$\|f\|_{M(\alpha)}=\sup_e\left\{(me)^{-\alpha}\int_e|f(x)|dx\right\},$$

where the supremum is over all measurable subsets of [0, 1]. If $\alpha=1$, $\Lambda(\alpha)=L$ and $M(\alpha)=M$. It is shown that these spaces are Banach spaces and $[\Lambda(\alpha)]^*=M(\alpha)$ although $[M(\alpha)]^*$ is not $\Lambda(\alpha)$. Various properties of these spaces are studied and applications are given to theorems involving Fourier series, fractional integration and moment problems.
R. E. Fullerton (Madison, Wis.).

Haefeli, Hans Georg, und Pellegrino, Franco. Die Reihe von Fantappiè und die Stetigkeit der analytischen nicht linearen Funktionale. Comment. Math. Helv. 23, 153–173 (1949).

Ce travail apporte une contribution à l'étude des fonctionnels analytiques $F[y(t)]$: (I) les fonctionnelles polynomiales sont continues; (II) toute fonctionnelle analytique bornée dans un voisinage d'un point $y_0(t)$ est continue en ce point. Dans un précédent article [mêmes Comment. 21, 225–246 (1948); ces Rev. 9, 515] les auteurs avaient déjà établi la continuité des fonctionnelles analytiques linéaires. Les démonstrations utilisent essentiellement le développement:

$$(1)\quad F[y(t)+\alpha\varphi(t)]$$
$$=F[y(t)]+\sum_{n=1}^\infty\frac{\alpha^n}{n!}F^{(n)}[y(t);\overset{*}{\alpha}_1,\cdots,\overset{*}{\alpha}_n]\varphi(\underset{*}{\alpha_1})\cdots\varphi(\underset{*}{\alpha_n})$$
$$=F[y(t)]+\sum_{n=1}^\infty\frac{\alpha^n}{n!}G_n{}^y[\varphi(t)].$$

La fonctionnelle mixte

$$F^{(n)}[y(t);\alpha_1,\cdots,\alpha_n]$$
$$=\frac{\partial^n}{\partial\epsilon_1\cdots\partial\epsilon_n}F\left[y_0(t)+\frac{\epsilon_1}{\alpha_1-t_1}+\cdots+\frac{\epsilon_n}{\alpha_n-t_n}\right]_{\epsilon_1=\cdots=\epsilon_n=0}$$

est la derivée (fonctionnelle) $n$ème de $F[y(t)]$. Rappelons

l'expression de la différentielle:

$$(2)\quad \delta F=\epsilon\left\{\frac{d}{d\epsilon}F[y_0(t)+\epsilon\varphi(t)]\right\}=\frac{\epsilon}{2i\pi}\int_C F'[y_0(t);\alpha]\varphi(\alpha)d\alpha$$
$$=\epsilon F'[y_0(t);\overset{*}{\alpha}]\varphi(\underset{*}{\alpha}).$$

Les produits de composition notés * résultent d'une intégration le long d'une courbe C du plan complexe; C, "séparatrice," sépare dans ce plan les singularités de $\varphi(\alpha)$ de celles de la fonctionnelle mixte qui précède. Les notations sont celles de L. Fantappiè [Atti Accad. Naz. Lincei. Mem. Cl. Sci. Fis. Mat. Nat. (6) 3, 453–683 (1930)]. Le développement (1) que les auteurs appellent série de Fantappiè se distingue de développements analogues de Volterra et de Fréchet par le mode de composition complexe.

Si les dérivées d'ordre supérieur à n disparaissent, $F[y(t)]$ est dite polynomiale de degré n. Les coefficients $G_n{}^y[\varphi(t)]$ sont des fonctionnelles homogènes de degré n portant sur $\varphi(t)$. On a $G_n{}^y[k\varphi(t)]=k^nG_n{}^y[\varphi(t)]$ et

$$G_n{}^{(n)}[\varphi(t);\alpha_1,\cdots,\alpha_n]=n!F^{(n)}[y(t);\alpha_1,\cdots,\alpha_n];$$

$G_n{}^y$ est une fonctionnelle polynomiale de degré n. L'étude de (1) fournit un "théorème de Liouville": si une fonctionnelle analytique entière est bornée sur son domaine d'existence, elle est constante.

Une majoration des dérivées de $F[y(t)]$ est ensuite obtenue sous la forme: $|G_n{}^y[\varphi(t)]|\leqq M_n\sigma^n$ où M_n est indépendant de $\varphi(t)$ et $|\varphi(t)|<\sigma$. Elle permet par une majoration de la série (1), d'établir le resultat (I). Les fonctionnelles $G_n{}^y[\varphi(t)]$ sont continues. Une série $\sum_n H_n[\varphi(t)]$ étant dite converger totalement dans Ω si max $|H_n[\varphi(t)]|\leqq H_n{}^*$ pour $\varphi(t)\epsilon\Omega$ et si $\sum_n H_n{}^*$ converge, les auteurs montrent que (a) si (1) converge totalement sur la frontière d'un voisinage de 0, elle converge totalement à l'intérieur; (b) si (1) converge totalement pour $y(t)$ appartenant à un voisinage de $y_0(t)$ et $|\varphi(t)|<\sigma$, $F[y(t)]$ est continue au point $y_0(t)$. Le résultat (II) en découle. Les dérivées successives satisfont en outre à $|F^{(n)}[y(t);\alpha_1,\cdots,\alpha_n]|\leqq Mk^nn!$, où $M=\max|F[y(t)]|$ dans le voisinage considéré. *P. Lelong* (Lille).

Yosida, Kôsaku. Integration of Fokker-Planck's equation in a compact Riemannian space. Ark. Mat. 1, 71–75 (1949).

Let $R=\{x\}$, $x=(x^1,\cdots,x^n)$ be an n-dimensional compact Riemannian space with a metric $ds^2=g_{ij}(x)dx^idx^j$. Let $L^1(R)$ be the L^1-space on R with respect to the measure defined by $dx=\{g(x)\}^{\frac{1}{2}}dx^1\cdots dx^n$, where $g(x)=\det(g_{ij}(x))$. It is shown that, under certain continuity assumptions on $a^i(x)$ and $b^{ij}(x)$, the Fokker-Planck equation:

$$\frac{\partial}{\partial t}f(t,x)=Af(t,x)\qquad(t\geqq0),$$

$$(Af)(x)=\{g(x)\}^{-\frac{1}{2}}\frac{\partial}{\partial x^i}[-\{g(x)\}^{\frac{1}{2}}a^i(x)f(x)]$$
$$+\{g(x)\}^{-\frac{1}{2}}\frac{\partial^2}{\partial x^i\partial x^j}[\{g(x)\}^{\frac{1}{2}}b^{ij}(x)f(x)]$$

with the initial condition $f(0,x)=f(x)\epsilon L^1(R)$, where $b^{ij}(x)$ is positive definite, has a solution $f(t,x)=U_tf(x)$, where $\{U_t:t\geqq0\}$ is a strongly continuous one-parameter semi-group of positive linear transformations of $L^1(R)$ which preserve the norm of positive elements of $L^1(R)$ satisfying strong $\text{limit}_{\delta\to0}\,\delta^{-1}(U_{t+\delta}-U_t)f=\tilde{A}U_tf$, where $\tilde{A}$ is the closed extension of A. The proof is based on a result of E. Hille [Functional Analysis and Semi-Groups, Amer. Math. Soc. Colloquium Publ., v. 31, New York, 1948; C. R. Acad. Sci. Paris 225, 445–447 (1947); these Rev. 9, 594, 193] and of the author [J. Math. Soc. Japan 1, 15–21 (1948); these Rev. 10, 462]. *S. Kakutani* (New Haven, Conn.).

Maak, Wilhelm. Almost periodic invariant vector sets in a metric vector space. Proc. Nat. Acad. Sci. U. S. A. 36, 208–210 (1950).

For the abstract almost periodic functions defined by the author [Abh. Math. Sem. Univ. Hamburg 16, 56–71 (1949); these Rev. 11, 327] the following basic theorem is derived. Given an arbitrary (closed) left modul M, consisting of abstract almost periodic functions, then M is the smallest closed modul containing all (finite) irreducible left-moduls in M. From this the approximation theorem is deduced. Given an arbitrary invariant closed modul U, consisting of abstract almost periodic functions, then U is the smallest closed modul containing all (finite) irreducible moduls in V (the vector space whose elements are the values of the abstract functions). Consequently the Parseval equation for abstract almost periodic function is proved, when axioms I–III of H. Weyl's paper [Amer. J. Math. 71, 178–205 (1949); these Rev. 10, 461, 856] are assumed. Axiom IV of Weyl's paper is shown to be superfluous in the derivation of the Parseval equation. *B. R. Gelbaum.*

Barbašin, E. A. On the theory of general dynamical systems. Učenye Zapiski Moskov. Gos. Univ. 135, Matematika, Tom II, 110–133 (1948). (Russian)

The topic of this paper is the same as that of Minkevič's [see the following review]. There is considerable overlapping between the two. *W. H. Gottschalk.*

Minkevič, M. I. Theory of integral funnels in dynamical systems without uniqueness. Učenye Zapiski Moskov. Gos. Univ. 135, Matematika, Tom II, 134–151 (1948). (Russian)

This paper contains proofs of results previously announced by the author [Doklady Akad. Nauk SSSR (N.S.) 59, 1049–1052 (1948); 60, 341–343 (1948); these Rev. 9, 449, 517]. *W. H. Gottschalk* (Philadelphia, Pa.).

Theory of Probability

Kappos, D. A. Zur mathematischen Begründung der Wahrscheinlichkeitstheorie. S.-B. Math.-Nat. Kl. Bayer. Akad. Wiss. 1948, 309–320 (1949).

Propaganda for looking at probability theory from the point of view of Boolean algebras. *P. R. Halmos.*

Gini, Corrado. Rileggendo Bernoulli. Metron 15, 117–132 (1949).

The author in question is James Bernoulli, the author of Ars Conjectandi, on which the author comments.

Bass, Jean, et Lévy, Paul. Propriétés des lois dont les fonctions caractéristiques sont $1/\text{ch}\,z$, $z/\text{sh}\,z$, $1/\text{ch}^2\,z$. C. R. Acad. Sci. Paris 230, 815–817 (1950).

The Fourier transforms of the three functions mentioned in the title are evaluated. The integrals in question are quite well known, in fact they are favourite class-room exercises for the calculus of residues, and they are also

contained in any good table of Fourier transforms [cf., for instance, G. A. Campbell and R. M. Foster, Fourier Integrals for Practical Applications, Bell Telephone Laboratory, New York, 1931, nos. 607.1, 614]. The coefficients in the power series expansions of the characteristic functions are expressed in terms of Bernoulli numbers, and the authors remark quite rightly that this expression is a simple consequence of the generating function for Bernoulli numbers.
A. Erdélyi (Pasadena, Calif.).

Armitage, P. An overlap problem arising in particle counting. Biometrika 36, 257–266 (1949).

Circular particles of diameter δ are distributed randomly in a plain domain. A "clump" is formed if several particles overlap, and each clump is counted as one unit. The author derives the mean number of counted units neglecting edge-effects. The formula is generalized to the case of unequal particles and also to rectangular particles. The point of departure is the distribution function for the distance between two points. *W. Feller* (Ithaca, N. Y.).

Chung, Kai Lai, and Feller, W. On fluctuations in coin-tossing. Proc. Nat. Acad. Sci. U. S. A. 35, 605–608 (1949).

Let $X_1, X_2, \cdots$ be independent random variables, each taking the value 1 with probability $\frac{1}{2}$, and write $S_n=X_1+\cdots+X_n$. It is agreed to call S_n "positive" if $S_n>0$ or if $S_n=0$ but $S_{n-1}>0$; otherwise S_n is called "negative." Let N_n denote the number of "positive" terms among $S_1, \cdots, S_n$. The most striking results of the paper are expressed by the relations

$$P(N_{2n}=2r)=2^{-2n}\binom{2r}{r}\binom{2n-2r}{n-r}$$

and $P(N_{2n}=2r|S_{2n}=0)=1/(n+1)$, which show that the character of the distribution of the random variable N_{2n} is radically changed by the introduction of the "tie" $S_{2n}=0$. In the usual interpretation in terms of coin tossing, the hypothesis $S_{2n}=0$ implies that the game is concluded at a moment when neither player has a gain or loss. Without this hypothesis, there is a high probability (for large n) that $N_{2n}/2n$ deviates considerably from $\frac{1}{2}$, i.e., that one party "leads" during a great proportion of the $2n$ first tosses. However, on the hypothesis $S_{2n}=0$, there is a uniform distribution of the possible values of $N_{2n}/2n$. *H. Cramér.*

***Doob, J. L. Application of the theory of martingales.** Le Calcul des Probabilités et ses Applications. Colloques Internationaux du Centre National de la Recherche Scientifique, no. 13, pp. 23–27. Centre National de la Recherche Scientifique, Paris, 1949.

For the theory of martingales cf. the author's paper [Trans. Amer. Math. Soc. 47, 455–486 (1940); these Rev. 1, 343] where the term "property ϵ" is used for martingales. The author now shows how the theory implies the strong law of large numbers for independent random variables with a common distribution. The main part of the paper is devoted to the problem of inverse probabilities. Assume that to different values of θ there correspond different distributions $F(x, \theta)$ (which are Baire functions of x and θ) and that θ is a random variable with distribution $G(\theta)$. It is shown that if θ is chosen in accordance with $G(\theta)$ and then an infinite sample of mutually independent observations $x_1, x_2, \cdots$ is taken, the (unknown) fixed value θ becomes a function of $x_1, x_2, \cdots$. *W. Feller.*

Cantelli, Francesco Paolo. Considerazioni sulla legge uniforme dei grandi numeri. I. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 6, 550–555 (1949).

The author resumes his studies of the strong law of large numbers [Atti Accad. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (5) 26_1, 39–45 (1917)] in such a way that he avoids the use of total probabilities. A sequence of trials on an event of constant probability p is considered; let v_i/i be the relative frequency of the occurrence of the event in i trials. Choose an arbitrary number $\eta>0$; then it is possible to determine an integer $n=n(\eta)$, depending on η, such that the probability of the simultaneous inequalities

$$(*)\quad |v_n/n-p|<n^{-\frac{1}{4}},\ |v_{n+1}/(n+1)-p|<(n+1)^{-\frac{1}{4}}, \cdots, \\ |v_N/N-p|<N^{-\frac{1}{4}}$$

exceeds $1-\eta$ for any fixed $N>n$. The author emphasizes that this formulation avoids the use of infinitely many inequalities (*). *E. Lukacs* (Washington, D. C.).

Cantelli, Francesco Paolo. Sulle probabilità di Karup. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 6, 397–402 (1949).

Karup studied populations subject to several causes of decrement and introduced the concepts of dependent and independent probabilities of elimination. A systematic presentation of these ideas was given by A. Loewy [S.-B. Heidelberger Akad. Wiss. Abt. A. Math.-Phys. Kl. 8, no. 6 (1917)]. Loewy as well as other authors discussing the subject assume that the probabilities of elimination are differentiable functions. The author modifies Karup's theory so as to describe the development of the population by functions of bounded variation. These functions, however, cannot be interpreted as distribution functions.
E. Lukacs (Washington, D. C.).

Ottaviani, Giuseppe. Intorno alle probabilità di Karup e legame con la teoria dei capitali accumulati. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 6, 679–685 (1949).

The author applies Cantelli's approach [see the preceding review] to the theory of capitalization. *E. Lukacs.*

Sapogov, N. A. General form of a limit theorem for independent random vectors. Doklady Akad. Nauk SSSR (N.S.) 70, 765–768 (1950). (Russian)

The author extends the standard necessary and sufficient conditions for the validity of the central limit theorem (for sums of mutually independent random variables) to vector-valued random variables. *J. L. Doob* (Ithaca, N. Y.).

***Fortet, R. Probabilité de perte d'un appel telephonique. Régime non stationnaire. Influence du temps d'orientation et du groupement des lignes.** Le Calcul des Probabilités et ses Applications. Colloques Internationaux du Centre National de la Recherche Scientifique, no. 13, pp. 105–113. Centre National de la Recherche Scientifique, Paris, 1949.

The author describes the method of treating counter problems proposed by the reviewer [Studies and Essays Presented to R. Courant, pp. 105–115, Interscience Publishers, New York, 1948; these Rev. 9, 294] and his own method for certain trunking problems [C. R. Acad. Sci. Paris 226, 159–161, 1502–1504 (1948); these Rev. 9, 361, 518]. He investigates Erlang's formula for losses and gives an approximation method for practical calculations.
W. Feller (Ithaca, N. Y.).

✶**Romanovskiĭ, V. I. Diskretnye Cepi Markova. [Discrete Markov Chains].** Gosudarstvennoe Izdatel'stvo Tehniko-Teoretičeskoĭ Literatury, Moscow-Leningrad, 1949. 436 pp.

Fundamental concepts and theorems; Indecomposable and decomposable acyclic chains C_n; Indecomposable cyclic chains C_n; Characteristic functions of discrete Markov chains; Correlation in chains; Markov-Bruns chains; Complex chains; Supplements and examples.

Table of contents.

Koopman, B. O. A generalization of Poisson's distribution for Markoff chains. Proc. Nat. Acad. Sci. U. S. A. **36**, 202–207 (1950).

For each n let $U_{n1}, \cdots, U_{nn}$ be random variables forming a two-state Markov chain with stationary transition probabilities. Suppose that the states are 0 and 1 and that when $n\to\infty$, $E\{\sum_k U_{nk}\}\to m$, $\max_k \Pr\{U_{nk}=1 \mid U_{n1}=0\}\to 0$, $\Pr\{U_{nk}=1 \mid U_{nk-1}=1\}\to a$, $\Pr\{U_{n1}=1\}\to p_1$. Then the most general limiting distribution of $\sum_k U_{nk}$ is found explicitly as a function of m, a, p_1. The corresponding result in the non-stationary case is stated without proof. *J. L. Doob.*

Rozenknop, I. Z. On some properties of collections of closed paths in a system with n states and given connections among them. Izvestiya Akad. Nauk SSSR. Ser. Mat. **14**, 95–100 (1950). (Russian)

This is another proof of a theorem proved by Čulanovskiĭ [Doklady Akad. Nauk SSSR (N.S.) **69**, 301–304 (1949); these Rev. **11**, 256]. *J. L. Doob* (Ithaca, N. Y.).

Rice, S. O. Communication in the presence of noise—probability of error for two encoding schemes. Bell System Tech. J. **29**, 60–93 (1950).

Let $X_0, \cdots, X_K, Y$ be $(2N+1)$-dimensional vector random variables whose $(K+2)(2N+1)$ components are mutually independent and normal, with zero means. The X_j components have a common variance and the Y components have a common variance. The author calculates the probability that $|X_j - X_0 - Y| > Y$, $j=1, \cdots, K$, and finds asymptotic expressions for this probability when $N, K\to\infty$. In the communications interpretation X_0 and Y correspond respectively to the transmitted signal and the noise. The above probability is the probability that none of K other signals will be mistaken for the signal sent. Two encoding systems are formulated which give approximately maximum rates of signalling in the presence of noise.

J. L. Doob (Ithaca, N. Y.).

Mathematical Statistics

✶**Mood, Alexander McFarlane. Introduction to the Theory of Statistics.** McGraw-Hill Book Company, Inc., New York, Toronto, London, 1950. xiv+433 pp. $5.00.

As its name implies, this book is intended as an introduction to statistics, the only prerequisite being a thorough year's course in calculus. No previous knowledge of probability is postulated, and the first part of the book is given over to an introduction to probability. The book is replete with interesting exercises many of which carry the theory further. An idea of the scope of the book can be obtained from the chapter titles, which are as follows: Probability and combinatorial methods, Discrete distributions, Distributions for continuous variates, Expected values and moments, Special continuous distributions, Sampling, Point estimation, The multivariate normal distribution, Sampling distributions, Interval estimation, Tests of hypotheses, Regression and linear hypotheses, Experimental designs and the analysis of variance, Sequential tests of hypotheses, Distribution-free methods. This book provides a competent introduction to modern statistics. It is an excellent textbook and teachers will welcome it. Naturally, the reviewer does not agree with all of the author's statements. The reviewer would, for example, challenge the statement on page 161 that the two papers cited there "virtually solved the whole problem of point estimation. . . ." This is stated simply to point up the differences of opinion and not to detract from the merits of this work. *J. Wolfowitz.*

Pompilj, Giuseppe. Teorie statistiche della significatività e conformità dei risultati sperimentali agli schemi teorici. Statistica, Milano **8**, 7–42 (1948).

Noack, Albert. A class of random variables with discrete distributions. Ann. Math. Statistics **21**, 127–132 (1950).

The author investigates discrete probability functions which may be represented by $p(x)=a_x s^x/f(s)$, $f(s)=\sum_{x=0}^{\infty} a_x s^x$, $x=0, 1, 2, \cdots$. He finds the mean, central moments, cumulants, characteristic functions, and recursion formulas for moments and cumulants of such probability functions. Various special cases illustrate the general theory such as the Poisson, the Pólya-Eggenberger distribution, and the hypergeometric distribution $F(a, b, c, s)$. *L. A. Aroian.*

Shenton, L. R. On the efficiency of the method of moments and Neyman's type A distribution. Biometrika **36**, 450–454 (1949).

Faleschini, Luigi. Su alcune proprietà dei momenti impiegati nello studio della variabilità, asimmetria e curtosi. Statistica, Milano **8**, 503–513 (1948).

Hemelrijk, J. On the determination of confidence intervals and estimates for the coefficients of a straight line from a number of inaccurately observed points. Math. Centrum Amsterdam. Rapport **ZW-1949-013**, 39 pp. (1949). (Dutch)

Expository article.

Plackett, R. L. A historical note on the method of least squares. Biometrika **36**, 458–460 (1949).

✶**Wishart, John. Test of homogeneity of regression coefficients, and its application in the analysis of covariance.** Le Calcul des Probabilités et ses Applications. Colloques Internationaux du Centre National de la Recherche Scientifique, no. 13, pp. 93–99. Centre National de la Recherche Scientifique, Paris, 1949.

Expository article.

Cavalli, Luigi L. Sulla correlazione media fra più caratteri in relazione alla biometria. Metron **15**, 173–188 (1949).

Grazia-Resi, Bruno. Nuove ricerche sugli indici di cograduazione fra serie con termini uguali. Statistica, Milano **8**, 409–426 (1948).

Salvemini, Tommaso. Nuovi procedimenti di calcolo degli indici di dissomiglianza e di connessione. Statistica, Milano 9, 3–26 (1 plate) (1949).

Krishna Iyer, P. V. The theory of probability distributions of points on a line. J. Indian Soc. Agric. Statistics 1, 173–195 (1948).

This is a comprehensive study which gives the first four cumulants of runs of various kinds of elements. Results conditional upon fixed numbers of elements of each kind (nonfree sampling) and unconditional results (free sampling) are given. The author calls attention to errors by Mood [Ann. Math. Statistics 11, 367–392 (1940); these Rev. 2, 228], and Mahalanobis [Philos. Trans. Roy. Soc. London. Ser. B. 231, 329–451 (1944)]. The author reports the following typographical errors: page 176, line 18: in place of $24(5m-11)p^5$ read $24(5m-11)p^6$; page 192, line 4: $T_2=m-n_1-2k$ −number of runs of length unity; page 194, line 5: in place of $(\sum n_r^4)$ read $(\sum n_r)^4$. *J. Wolfowitz.*

Krishna Iyer, P. V. Difference equations of moment-generating functions for some probability distributions. Nature 165, 370 (1950).

The author gives difference equations satisfied by the moment generating functions of the distributions of various kinds of runs discussed in the paper reviewed above. The statements about asymptotic normality made there are thus rigorously proved. *J. Wolfowitz* (New York, N. Y.).

Gini, Corrado. Le medie dei campioni. Metron 15, 13–28 (1949).

The author considers finite populations and defines generalized mean values. He establishes a number of simple theorems on these means and studies in particular the formation of means of the sample means and their relation to the mean of the population. *E. Lukacs.*

Hammersley, J. M. The unbiassed estimate and standard error of the interclass variance. Metron 15, 189–205 (1949).

Given a one-way classification with unequal class-numbers, the author derives the formula for the unbiased estimate of the interclass variance and computes the sampling variance of this estimate. Given an upper limit for the total number of observations to be taken, the number of classes and number of observations in each class which will minimize the variance of the estimate are determined. A numerical example is discussed. *G. E. Noether* (New York, N. Y.).

Pillai, K. C. S. On the distributions of midrange and semi-range in samples from a normal population. Ann. Math. Statistics 21, 100–105 (1950).

Let $x_1, \cdots, x_n$ be an ordered sample from a normal population with zero mean and unit variance. The author finds the joint distribution of $M=\frac{1}{2}(x_1+x_n)$, the midrange, and $W=\frac{1}{2}(x_n-x_1)$, the semi-range, and the distributions of M and of W. The results are given in a form suitable for numerical calculations. *L. A. Aroian.*

Link, Richard F. The sampling distribution of the ratio of two ranges from independent samples. Ann. Math. Statistics 21, 112–116 (1950).

Massey, Frank J., Jr. A note on the estimation of a distribution function by confidence limits. Ann. Math. Statistics 21, 116–119 (1950).

Let $X_1, \cdots, X_n$ be mutually independent random variables with a continuous distribution $F(x)$, and let $P_n(\lambda)$ be the probability that the empirical distribution of the sample deviates from $F(x)$ by at most $\lambda n^{-\frac{1}{2}}$. The $P_n(\lambda)$ can be calculated from recursive linear equations. The author gives the results of a calculation for $n=10(10)80$ and $\lambda=.9(.1)1.4$. For $n=80$ the distribution is very close to Kolmogorov's limiting form. *W. Feller* (Ithaca, N. Y.).

Gumbel, E. J., and Keeney, R. D. The geometric range for distributions of Cauchy's type. Ann. Math. Statistics 21, 133–137 (1950).

The authors consider symmetric distributions with the asymptotic behaviour (1) $F(x)=1-Ax^{-k}(1+\epsilon(x))$ $(x\to\infty)$. In order to estimate the constants A, k they introduce the so-called geometric range $\rho=(-x_1x_n)^{\frac{1}{2}}$, where x_1, x_n signify the smallest and the largest observation in a sample of n. The asymptotic distribution of the variable $\xi=2n\{F(x_1)(1-F(x_n))\}^{\frac{1}{2}}$ is known [Elfving, Biometrica 34, 111–119 (1947); these Rev. 8, 395] and the same for any symmetric continuous distribution. In the present case, according to (1), $\xi=2nA\rho^{-k}$ for large n. Due to this fact, the parameters may be estimated from a sample of observed geometric ranges, obtained from sufficiently large subsamples. This can be done either by means of the two first moments of $1/\rho$; or by means of two appropriate quantiles of the empirical ρ-distribution; or, finally, using a probability paper constructed by the authors for that purpose. *G. Elfving* (Ithaca, N. Y.).

Gjeddebæk, N. F. Contribution to the study of grouped observations. Application of the method of maximum likelihood in case of normally distributed observations. Skand. Aktuarietidskr. 32, 135–159 (1949).

The author points out that in the case of grouped measurements the computed sample mean and variance are not consistent estimates of the mean and variance of the population even after Sheppard's corrections are applied. He derives the maximum likelihood estimates of the mean and variance of the population and estimates of the covariance of these estimates for the case where the population is known to be normal. A relatively simple method of computation is outlined making use of tables constructed for this problem. An example involving very coarse grouping is presented. *H. Chernoff* (Urbana, Ill.).

Hyrenius, Hannes. Sampling distributions from a compound normal parent population. Skand. Aktuarietidskr. 32, 180–187 (1949).

Let $N(x)=\sum_{i=1}^{n}p_i\varphi(x, a_i, \lambda_i)$, the compound normal curve, where $p_i>0$, $\sum p_i=1$, $\varphi(x, a_i, \lambda_i)$ the normal probability function with mean a_i and variance λ_i. The characteristic functions of $\sum x/N$ and $\sum x^2/N$ in samples of N from $N(x)$ are derived. The major part of the paper treats the special case $N_1(x)$, $n=2$, a_1, a_2, and $\lambda_1=\lambda_2=\lambda$. For $N_1(x)$ the author derives the joint distribution and the characteristic function of the sample mean and variance, the distribution of the ratio of two sample variances, and what amounts to a Student's t-distribution. The symbols are not always explicitly defined and the conciseness of the paper makes reading difficult. *L. A. Aroian* (Culver City, Calif.).

Box, G. E. P. A general distribution theory for a class of likelihood criteria. Biometrika **36**, 317–346 (1949).

Suppose that the moments of a statistic W used in testing some hypothesis are given by

$$E(W^h)=c\left\{\frac{\prod_{j=1}^{k}(y_j^{y_j})}{\prod_{i=1}^{m}(x_i^{x_i})}\right\}^h \frac{\prod_{i=1}^{m}\{\Gamma\{x_i(1+h)+\xi_i\}\}}{\prod_{j=1}^{k}\{\Gamma\{y_j(1+h)+\eta_j\}\}},$$

where $\sum_{i=1}^{m}x_i=\sum_{j=1}^{k}y_j$. For a number of hypotheses in multivariate analysis, the likelihood ratio or some power of it has moments of the above form, e.g., (1) equality of variances and covariances from k samples, (2) independence of k groups of variates, (3) equality of means, variances and covariances from k samples. The quantities x_i, y_j depend on the number of variates, number of samples, and number of degrees of freedom in each sample. Let $M=-2\log W$, ρ be any constant not exceeding 1. The characteristic function $\Phi(t)$ of ρM can easily be found from the above expression of the moments of W; let $\Psi(t)=\log\Phi(t)$. An asymptotic expansion for the terms of the form $\log\Gamma(x+h)$ can be found by a generalization of Stirling's formula, yielding an asymptotic expansion for $\Psi(t)$. Then, by taking antilogarithms and expanding exponential terms in Taylor's series, an asymptotic expansion for $\Phi(t)$ is found. The Fourier transform then yields an expansion for the cumulative distribution of ρM in terms of χ^2-distributions of successively greater degrees of freedom. The constant ρ can be chosen so as to make the convergence of the expansion more rapid. Other approximations for the distribution of M are found. Ignoring terms of order x_i^{-2}, y_j^{-2}, M is distributed as $C\chi^2$ for suitable C. Ignoring terms of order x_i^{-3}, y_j^{-3}, a function of M is distributed as the variance ratio. The tests for hypotheses (1) and (2) are considered in greater detail.

K. J. Arrow (Stanford University, Calif.).

Moore, P. G. A test for randomness in a sequence of two alternatives involving a 2×2 table. Biometrika **36**, 305–316 (1949).

The author continues the work of F. N. David [Biometrika **34**, 335–339 (1947); these Rev. **9**, 600]. After suitable reduction of the initial problem the final problem is as follows. There is given a sequence of stationary binomial chance variables, and it is desired to test the hypothesis of complete stochastic independence against the alternative of a Markov chain. The test discussed is based on the number of runs. The author obtains the asymptotic conditional power function, the condition being that the number of each of the two values which the chance variables can assume is fixed. Illustrative examples are discussed. *J. Wolfowitz.*

Sillitto, G. P. Note on approximations to the power function of the "2×2 comparative trial." Biometrika **36**, 347–352 (1949).

The paper gives numerical comparisons of two types of approximation to the power function of a well-known statistical test for whether two binomial parameters, P_1, P_2, are equal. One type has been presented by Patnaik [Biometrika **35**, 157–175 (1948); these Rev. **9**, 603]. The other type of approximation involves use of the angular transformation of a binomial chance quantity. Specifically, the comparisons are for the "two-sided" test with significance levels .02, .10 and the following pairs of sample sizes: (18, 12), (15, 15), (30, 30). In most of the cases the angular-transformation approximation is closer to the true power. The author also considers the estimation of sample size when the power is given. *D. F. Votaw, Jr.* (New Haven, Conn.).

David, F. N. Note on the application of Fisher's k-statistics. Biometrika **36**, 383–393 (1949).

The author reviews the theory of cumulants as developed and extended by R. A. Fisher, J. Wishart, and M. G. Kendall. The results are used in the investigation of the approximate distribution of the coefficient of variation under the assumption that the parent population from which the sample is drawn is a type A Gram-Charlier series of three terms. The first four central moments of the coefficient of variability are determined to the order of n^{-2}, where n is the sample size. *L. A. Aroian* (Culver City, Calif.).

David, F. N. The moments of the z and F distributions. Biometrika **36**, 394–403 (1949).

Let π_1 and π_2 be any two populations possessing cumulants of a specified order, s_1^2 and s_2^2, the sample estimates of the population variances σ_1^2 and σ_2^2. It is assumed that the two samples x_i, x_j, $i, j=1, 2, \cdots, n$, are independently drawn from π_1 and π_2, respectively. Let $z=\frac{1}{2}\log(s_1^2/s_2^2)$ and $F=e^{2z}$. The author finds the mean and essentially the first four cumulants of z and F approximately, including terms of order n^{-2}, without the assumption of normality of π_1 and π_2. The results are applied to investigate the effects of skewness on z if both π_1 and π_2 are assumed to be the same type A Gram-Charlier series. Some of the historical references are inaccurate. [The notation is that of the reviewer.]

L. A. Aroian (Culver City, Calif.).

Gayen, A. K. The distribution of 'Student's' t in random samples of any size drawn from non-normal universes. Biometrika **36**, 353–369 (1949).

Let $f(x)$, the parent population, be specified by

$$(*)\quad f(x)=\varphi(x)-\lambda_3\varphi^{(3)}(x)/3!+\lambda_4\varphi^{(4)}(x)/4!+10\lambda_3^2\varphi^{(6)}(x)/6!,$$

where $\varphi(x)=(2\pi)^{-\frac{1}{2}}e^{-x^2/2}$, $\varphi^{(\nu)}(x)=d^\nu\varphi(x)/dx^\nu$; λ_3 and λ_4 are the measures of universal skewness and kurtosis. The author derives the exact distribution of Student's t, $t=(\bar{x}-m)n^{\frac{1}{2}}/s$, where $\bar{x}$ is the sample mean, s the sample standard deviation and n the number of degrees of freedom. Included in the course of the work is the distribution of s^2 from (*). The results are extended to obtain Student's t-distribution from (*') obtained by adding terms involving λ_5, $\lambda_3\lambda_4$, and λ_3^3 to (*). The author infers that the effect of λ_4 is rather small, but that the effects of the λ_3 and λ_3^2 terms are rather serious in a one-sided test as compared with the normal theory; and in the case of a two-sided test of significance that the effect of λ_3^2 remains. Extensive tables are given for sample sizes corresponding to degrees of freedom 2(1)6, 8, 12, 24 and ∞. Examples and comparisons with experimental results conclude the paper. *L. A. Aroian.*

Ghurye, S. G. On the use of Student's t-test in an asymmetrical population. Biometrika **36**, 426–430 (1949).

An approximate expression is found for the power of Student's t-test when used to test a one-sided hypothesis concerning the mean of a nonnormal population where the critical region has been determined on the erroneous assumption of normality in the parent population. The nonnormal population considered differs only from normality in having a nonzero third cumulant; in the derivation the author assumes the standardized third cumulent is "sufficiently small." Subject to this restriction the change in the power of the test from the normal case is negligible.

D. G. Chapman (Seattle, Wash.).

Nair, K. R. Certain symmetrical properties of unbiased estimates of variance and covariance. J. Indian Soc. Agric. Statistics 1, 162–172 (1948).

Starting with the well-known fact that the variance of a set of values of a variate is one-half the mean square of all possible variate differences and that a similar relation holds for covariances of a bivariate sample, after some discussion the author proceeds to generalizations as follows. (1) Let the set of values $x_1, \cdots, x_n$ be arranged in order of magnitude and let a set of coefficients l_i, $i=1, \cdots, n$, of sum zero and sum of squares unity be also arranged in order of magnitude. Then sums of the form $\sum_{i,j=1}^{n} l_i x_j$ in which i and j take each of the values $1, \cdots, n$ once and once only are called general linear contrasts of $x_1, \cdots, x_n$. (2) Let x_{ij}, $i=1, \cdots, p$; $j=1, \cdots, q$, be a set of pq x's and l_{hk}, $h=1, \cdots, p$; $k=1, \cdots, q$, a set of pq coefficients whose sums with respect to h or k are each zero and whose sum of squares is unity. If the x_{hk}'s are arranged in a $p \times q$ table and the l_{hk}'s are likewise arranged, the sum of products $\sum l_{hk} x_{ij}$ of elements in corresponding cells is called a general interaction contrast. Properties of both are discussed and in particular standard test variances are shown to be derivable from the universe of these contrasts. *C. C. Craig* (Ann Arbor, Mich.).

***Halphen, E. Quelques remarques sur le problème de l'estimation.** Le Calcul des Probabilités et ses Applications. Colloques Internationaux du Centre National de la Recherche Scientifique, no. 13, pp. 87–91. Centre National de la Recherche Scientifique, Paris, 1949.

The principal point of the author is that, in any problem of estimation, the goal to be achieved should be set by the special circumstances of the problem. A number of examples are discussed to illustrate this point.

J. Wolfowitz (New York, N. Y.).

Finney, D. J. The estimation of the parameters of tolerance distributions. Biometrika 36, 239–256 (1949).

Suppose that each member of a population has a tolerance u with respect to some stimulus, i.e., a reaction takes place at doses exceeding u and u has a distribution with density $\sigma^{-1}f((u-\mu)/\sigma)$, where $f(v)$ is a specified density with mean 0 and variance 1. It is desired to estimate μ and σ and other related parameters. It is assumed that a certain percentage of the population will react and a certain percentage will not react whether or not a dose is given. (These percentages may be known in advance or estimated with the other parameters depending on the particular problem.) The maximum likelihood estimates are derived. An iterative method is applied to compute the estimates. The method is applied in several examples. *H. Chernoff.*

Radhakrishna Rao, C. On some problems arising out of discrimination with multiple characters. Sankhyā 9, 343–366 (1949).

In discriminating between two populations, the question arises whether or not adding an additional q characteristics to p already used will increase the distance (in Mahalanobis's sense) between the populations and therefore reduce the frequency of misclassification. Let n_1 and n_2 be the sample sizes from the two populations, $c=n_1n_2/(n_1+n_2)$, and D_p^2 the sample distance of the two populations based on p characteristics. Define

$$M_p=1+[c/(n_1+n_2-2)]D_p^2, \quad R=M_p/M_{p+q}.$$

Let β^2 be the square of the true distance between the populations based on p characteristics and $\alpha^2+\beta^2$ the square of the distance between the two populations based on $p+q$ characteristics. The author finds the conditional distribution of R given $S=1/M_p$, the joint distribution of R and S, and the unconditional distribution of R, for any α and β. Since the first distribution does not involve β, while the distribution of S does not involve α, the author proposes to test the hypothesis $\alpha=0$ by the use of R, with S as an ancillary statistic, in the case where the population value of β is unknown. When β is known to be 0, the author considers the alternative test statistic $w=M_{p+q}+M_p$, and derives its distribution; there is reason to believe that w is a more efficient test than R under these circumstances.

From the power function of the D^2-statistic, the author finds that (1) the power of the test is maximized, for a given number of characters and total sample size, by equating n_1 and n_2; (2) if the square of the distance increases proportionately to the number of characters, increasing this leads to no loss of efficiency except in a very small sample. If β is known to be 0 it may still be that the test for α based on R is more efficient than a test based on the last q characters alone; this will certainly be so if the two samples are so chosen that the sample distance based on the first p characters is made sufficiently small. The author concludes by considering unbiased estimation of α^2 and β^2, and some computational problems. *K. J. Arrow.*

***Delaporte, P. Sur une utilisation systématique de la statistique mathématique en analyse factorielle.** Le Calcul des Probabilités et ses Applications. Colloques Internationaux du Centre National de la Recherche Scientifique, no. 13, pp. 101–104. Centre National de la Recherche Scientifique, Paris, 1949.

Let ${}_jX_i$ be the score of the jth individual on the ith test $(i=1, \cdots, k)$, and let ${}_jx_i=[{}_jX_i-E(X_i)]/\sigma(X_i)$. It is assumed that ${}_jx_i=\alpha_i\,{}_jG+\beta_i\,{}_jB+\cdots+\lambda_i\,{}_jS_i$, where $\alpha_i, \beta_i, \cdots, \lambda_i$ are factor loadings, ${}_jG$ is the normalized score of the jth individual on the general factor, ${}_jB, \cdots$ on "group factors," and ${}_jS_i$ on the ith specific factors. If the loadings of the group factors are 0, $\rho_{gi}/\rho_{hi}=\rho_{gj}/\rho_{hj}$ for all $i, j \neq g, h$ and for each pair $g \neq h$, where $\rho_{gi}=Ex_gx_i$. The author suggests rejecting the hypothesis that group factor loadings are 0 if for any pair $g \neq h$ the confidence intervals for the $k-2$ ratios ρ_{gi}/ρ_{hi} (based on the sample correlations r_{gi}) do not have a common part. If the hypothesis is known to be false, similar methods can be used to test the hypothesis that only one group factor has nonzero loadings or to locate the nonzero loadings of group factors. If the group factor loadings are zero for two of the three tests h, i and j, $\alpha_h^2=\rho_{hi}\rho_{hj}/\rho_{ij}$ $(i, j \neq h, i \neq j)$. To estimate α_h^2, the author repeats his suggestion [C. R. Acad. Sci. Paris **222**, 525–527 (1946); these Rev. 7, 463] of using $[\sum_{i,j} r_{hi}r_{hj}/(r_{ij}W_{h\cdot ij})]/\sum_{i,j}W_{h\cdot ij}$, where $W_{h\cdot ij}$ are certain weights and the sum is extended over all permissible $i \neq j$. The methods can be extended to estimation of β_h^2, etc. *T. W. Anderson* (New York, N. Y.).

Delaporte, Pierre. Une condition nécessaire que les observations doivent remplir pour être représentables par un schéma d'analyse factorielle de Spearman. C. R. Acad. Sci. Paris 229, 973–975 (1949).

For the model treated in the preceding paper the author suggests that a confidence interval for α_h^2 can be obtained by use of $r_{hi}r_{hj}/r_{ij}$ for any pair $i, j \neq h$, $i \neq j$ assuming this ratio approximately normally distributed. The pairs i, j to be used in the point estimate for α_h^2 given in the preceding review are those pairs for which the confidence intervals have a common part. *T. W. Anderson* (New York, N. Y.).

Anscombe, F. J. Tables of sequential inspection schemes to control fraction defective. J. Roy. Statist. Soc. Ser. A. 112, 180–206 (1949).

Tables are given for the evaluation of the operating characteristic and average sample number of various sequential inspection schemes. The non-rectifying schemes (and notation) are due to Barnard [Suppl. J. Roy. Statist. Soc. 8, 1–21 (1946); these Rev. 8, 395], the formulae, on which the tables are based, having been given by Burman [ibid., 98–103 (1946); these Rev. 8, 395] and Stockman and Armitage [ibid., 104–112 (1946); these Rev. 8, 396]. The tables for the rectifying schemes are based on results by the author [ibid., 216–222 (1946); these Rev. 9, 49]. The choice of an inspection scheme is discussed from a general point of view and in the light of these tables. Some limiting formulae are stated or derived; for example, the probability of acceptance when the inspection scheme does not permit rejection. [This result is an immediate consequence of Markov's solution of the problem of the ruin of the gambler; cf. J. V. Uspensky, Introduction to Mathematical Probability, McGraw-Hill, New York-London, 1937, pp. 143–146.]

D. G. Chapman (Seattle, Wash.).

Kemperman, J. H. B. Some methods from sequential analysis. Math. Centrum Amsterdam. Rapport **ZW-1949-009**, 9 pp. (1949). (Dutch)

Kendall, M. G. Tables of autoregressive series. Biometrika 36, 267–289 (1949).

The author tabulates 16 empirical time series (to a minimum of 240 terms). The time series are solutions of $au_{t+1}+bu_t=\epsilon_{t+1}$ and $au_{t+2}+bu_{t+1}+cu_t=\epsilon_{t+2}$, with specified coefficients. The ϵ_t's are either mutually independent with a common rectangular or normal distribution, obtained from tables of random numbers, or are themselves the u_t's of a previously obtained series. *J. L. Doob.*

Strecker, Heinrich. Die Quotientenmethode, eine Variante der "Variate Difference" Methode. Mitteilungsblatt Math. Statist. 1, 115–130 (1 plate) (1949).

The author concerns himself with the trend analysis of certain time-series [mainly from economic or sociological data] where trend and disturbance are positively correlated in magnitude. To allow for this, he suggests the fitting of a scheme $y_i=m_i(1+x_i)$, where x_i is a disturbance variable of a stationary non-autocorrelated nature, and m_i the trend factor. If successive differences are taken of $(\log y_i)$, the trend component $(\log m_i)$ will then under certain assumptions be eventually eliminated, or in other cases weakened. By applying a least squares procedure to the logarithmic equation, the author also obtains approximate estimates of m_i in terms of the y-values. He shows finally that various other schemes allowing correlation between trend and disturbance are approximately reducible to the above form, $y_i=m_i(1+x_i)$. *P. Whittle* (Uppsala).

Williams, E. J. Experimental designs balanced for the estimation of residual effects of treatments. Australian J. Sci. Research. Ser. A. 2, 149–168 (1949).

The author considers a series of n treatments, each of which is applied to the same unit of experimental material. In such a case a treatment may affect the unit of material used and thus the response to subsequent treatments. It is also assumed that the position of the treatment in the series affects the response so that each treatment must occur equally often in every position. The author proposes designs which are in addition balanced for residual effects of preceding treatments. Three types are discussed. (1) Designs balanced for the immediately preceding treatment. Each treatment must be preceded equally often by each other treatment. (2) Designs balanced for any number of preceding treatments. (3) Designs balanced for the two immediately preceding treatments and their interaction. In this type each ordered triple of treatments must occur equally often. Designs of type (1) can be constructed by finding permutations $r_1r_2\cdots r_n$; $s_1s_2\cdots s_n$; $\cdots$ of the residues mod n such that among the differences r_i-r_{i+1}, s_j-s_{j+1}, $\cdots$ $(i, j=1, \cdots, n-1)$ each nonzero residue arises equally often. Only one permutation is necessary if n is even, two if n is odd. Each permutation is then taken as the first row of a Latin square, the other rows being obtained by adding $1, 2, \cdots, n-1$ to each digit. Designs of type 2 can be constructed from sets of orthogonal Latin squares. The author also announces solutions for the case $n=6$ and $n=10$, to be published later. Designs of type 3 are constructed for the case that n is a prime and for $n=4$, 8, 9. No general method for $n=p^\alpha$ (p a prime) has as yet been found. The author also gives explicitly the analysis of variance for designs of type 1 together with a numerical example.

H. B. Mann (Berkeley, Calif.).

Seal, H. L. Mortality data and the binomial probability law. Skand. Aktuarietidskr. 32, 188–216 (1949).

Mathematical Biology

***Malécot, G. Les processus stochastiques de la génétique.** Le Calcul des Probabilités et ses Applications. Colloques Internationaux du Centre National de la Recherche Scientifique, no. 13, pp. 121–126. Centre National de la Recherche Scientifique, Paris, 1949.

The author considers a population with non-overlapping generations. He studies the composition of successive generations with respect to a particular gene which can assume r different forms. Assuming arbitrary coefficients of mixing, selection, etc., one is in principle led to a general Markov chain. Under the usual simplifying assumptions one is led to a Fokker-Planck equation [cf. S. Wright, Proc. Nat. Acad. Sci. U. S. A. 31, 382–389 (1945); these Rev. 7, 319]. The author gives the stable solutions found by Wright and then studies in some detail the problem of migration without selection. *W. Feller* (Ithaca, N. Y.).

Skellam, J. G. The probability distribution of gene-differences in relation to selection, mutation, and random extinction. Proc. Cambridge Philos. Soc. 45, 364–367 (1949).

The functional equation $\phi(t)=e^{\lambda(t-1)}(e^{c(t-1)})$ provides the condition that the distribution of gene frequencies shall be constant from generation to generation, where λ is the mutation rate per generation and c is the survival value of

the gene. A close approximation to the solution is given by the negative binomial. *C. P. Winsor* (Baltimore, Md.).

Castellano, Vittorio. Sulle mutue relazioni tra i vari metodi per la determinazione della frequenza dei geni nei gruppi sanguigni. Metron 15, 375–401 (1949).

A comparative study of the various method of estimating empirical gene frequencies from observations.

W. Feller (Ithaca, N. Y.).

Harris, H., and Smith, C. A. B. The sib-sib age of onset correlation among individuals suffering from a hereditary syndrome produced by more than one gene. Ann. Eugenics 14, 309–318 (1949).

A theoretical discussion of the problem described in the title, with a discussion of the conditions under which bimodality may arise from the mixture of Gaussian populations.

C. P. Winsor (Baltimore, Md.).

TOPOLOGY

Seifert, Herbert, and Threlfall, William. Old and new results on knots. Canadian J. Math. 2, 1–15 (1950).

A carefully reasoned development of the central part of knot theory (the group of a knot, the Alexander polynomial, the homology groups and linking invariants of the cyclic coverings) together with explanation of new results by Graeub, Schubert, Seifert and Threlfall, this survey should furnish an excellent guide to the nonspecialist. The combinatorial point of view prevails and the underlying unsolved topological problems are not discussed. Although complete coverage was not attempted, the reviewer notes the lack of mention of the "problem of asphericity" [cf. J. H. C. Whitehead, Fund. Math. 32, 149–166 (1939)].

R. H. Fox (Princeton, N. J.).

Threlfall, William. Knotengruppe und Homologieinvarianten. S.-B. Heidelberger Akad. Wiss. Math.-Nat. Kl. 1949, no. 8, 357–370 (1949).

Let F be a finitely presented group whose commutator quotient group F/F' is infinite cyclic. Invariantly associated with F and a choice of generator x of F/F' is an equivalence class $\mathfrak{M}(x)$ of integral polynomial matrices; each presentation of F by (a finite number of) generators and relations determines some matrix $M(x)$ of the class $\mathfrak{M}(x)$. Let F_g denote the normal subgroup of F of index g that contains F'. The Betti number and the torsion coefficients of the commutator quotient group H_g of F_g are the rank defect and the elementary divisors of any one of an equivalence class $\mathfrak{M}_g$ of integral matrices. If P_g denotes the permutation matrix of the substitution $(12\cdots g)$ and $M(x)\epsilon\mathfrak{M}(x)$ then, as was demonstrated (in a special case) by Alexander [Trans. Amer. Math. Soc. 30, 275–306 (1928)], $M(P_g)\epsilon\mathfrak{M}_g$.

Let k be a simple closed polygon in spherical 3-space S. If F is the fundamental group of $S-k$ then H_g is the 1-dimensional homology group of the g-fold cyclic covering of $S-k$. Seifert [Math. Ann. 110, 571–592 (1934)] has shown that, given in S any orientable surface spanning k, there may be found a $2h\times 2h$ integral matrix Γ, where h is the genus of the surface, such that $E-\Gamma+x\Gamma'\epsilon\mathfrak{M}(x)$. The author now constructs, given any projection of k, an $(n-1)\times(n-1)$ integral matrix Γ, where n is the number of double points of the projection, such that again one has $E-\Gamma+x\Gamma'\epsilon\mathfrak{M}(x)$. [There is a slight confusion between the notation of this paper and that of Seifert's.] In either case a simple calculation made by Seifert [ibid.] shows that $\Gamma^g-(\Gamma-E)^g\epsilon\mathfrak{M}_g$.

As the author points out, this method of calculating H_g is more practical than the original method because the order of $M(P_g)$ increases with g while that of $\Gamma^g-(\Gamma-E)^g$ remains constant. [On the other hand Seifert's algorithm is the most practical of all since in general $2h$ will be much smaller than $n-1$. The example of the trefoil knot, chosen by the author to illustrate the new algorithm, is a little misleading because it is of genus one and has a projection with just three double points so that $2h=n-1$ in this case.]

R. H. Fox (Princeton, N. J.).

★**Pontryagin, L. S. Osnovy Kombinatornoĭ Topologii. [Foundations of Combinatorial Topology].** OGIZ, Moscow-Leningrad, 1947. 143 pp.

Complexes and their Betti groups, The invariance of the Betti groups, Continuous mappings and fixed points.

Table of contents.

Dowker, C. H. Čech cohomology theory and the axioms. Ann. of Math. (2) 51, 278–292 (1950).

L'auteur démontre que la théorie de cohomologie de Čech satisfait aux axiomes de Eilenberg et Steenrod. A ce but il lui faut generaliser la notion de cohomologie relative de Čech. *H. Freudenthal* (Utrecht).

Dugundji, J. A topologized fundamental group. Proc. Nat. Acad. Sci. U. S. A. 36, 141–143 (1950).

Using the topologized fundamental group χ of the connected and locally connected space R, the author constructs covering spaces corresponding to the open subgroups of χ. In the case of a closed subgroup the fibers become homeomorphic to the coset space. *H. Freudenthal* (Utrecht).

MacLane, Saunders, and Whitehead, J. H. C. On the 3-type of a complex. Proc. Nat. Acad. Sci. U. S. A. 36, 41–48 (1950).

Two complexes K and L are said to be of the same n-type if there exist maps $\varphi:K^n\to L^n$ and $\varphi':L^n\to K^n$ (where K^n and L^n are the n-skeletons of K and L) such that the maps $\varphi'\varphi|K^{n-1}:K^{n-1}\to K^{n-1}$ and $\varphi\varphi'|L^{n-1}:L^{n-1}\to L^{n-1}$ are both homotopic to identity maps. Any two connected complexes are of the same 1-type, and two complexes are of the same 2-type if and only if their fundamental groups are isomorphic. In this paper the topological problem of determining whether two given complexes are of the same 3-type is reduced to an algebraic problem. Included in this result is the reduction of the topological problem of classifying 2-dimensional complexes according to homotopy type to an algebraic problem.

An algebraic 3-type $T=(\pi_1,\pi_2,\mathbf{k})$ is defined to be a triple consisting of a multiplicative group π_1, an additive Abelian group π_2 which admits π_1 as a group of operators, and a 3-dimensional cohomology class $\mathbf{k}\epsilon H^3(\pi_1,\pi_2)$. The algebraic 3-type of a complex K is the triple $T(K)=(\pi_1(K),\pi_2(K),\mathbf{k}(K))$ consisting of the fundamental group, the second homotopy group of K with the usual operators of $\pi_1(K)$ on $\pi_2(K)$, and the invariant $\mathbf{k}^3=\mathbf{k}(K)$ [for the definition of $\mathbf{k}^3$, see S.

MacLane, Ann. of Math. (2) **50**, 736–761 (1949); these Rev. **11**, 415 or S. Eilenberg and S. MacLane, Proc. Nat. Acad. Sci. U. S. A. **32**, 277–280 (1946); these Rev. **8**, 398]. If $T=(\pi_1, \pi_2, \mathbf{k})$ and $T'=(\pi_1', \pi_2', \mathbf{k}')$ are algebraic 3-types, a homomorphism $\theta: T\to T'$ consists of a pair of homomorphisms $\theta_1:\pi_1\to\pi_1'$ and $\theta_2:\pi_2\to\pi_2'$ which are required to satisfy certain rather natural conditions. Then θ is called an isomorphism onto if both θ_1 and θ_2 are isomorphisms onto. The main results are the following. (1) Two complexes K and K' are of the same 3-type if and only if $T(K)$ and $T(K')$ are isomorphic. (2) Given any algebraic 3-type T, there exists a complex K such that $T(K)$ is isomorphic to T. (3) Given two complexes K and K' and a homomorphism $\theta=(\theta_1, \theta_2): T(K)\to T(K')$, there exists a continuous map $\varphi: K\to K'$ which induces the homomorphisms θ_1 and θ_2 provided that dimension $K\leqq 3$. Statement (1) follows directly from (3) and a previous result of J. H. C. Whitehead [Bull. Amer. Math. Soc. **55**, 213–245 (1949), theorem 2; these Rev. **11**, 48]. The main part of the paper is devoted to the proof of statements (2) and (3). *W. S. Massey.*

Specker, Ernst. Die erste Cohomologiegruppe von Überlagerungen und Homotopie-Eigenschaften dreidimensionaler Mannigfaltigkeiten. Comment. Math. Helv. **23**, 303–333 (1949).

The well-known fact that the fundamental group $\mathfrak{G}$ of a (finite connected) complex K determines the first cohomology group is generalized as follows: if $\bar{K}$ is a covering complex of K, belonging to the subgroup $\mathfrak{H}$ of $\mathfrak{G}$, then the first cohomology group B_1 of $\bar{K}$ (finite cochains) is determined (as operator group under the deck transformations) by the inclusion $\mathfrak{H}\subset\mathfrak{G}$. Every element $a\in\mathfrak{G}$ determines in a natural way a 1-chain $c_1(a)$ of $\bar{K}$, modulo boundaries, such that $c_1(a)$ covers a. Every 1-cocycle C_1 of $\bar{K}$ determines through the formula $KI(C_1, c_1(a))$ a function from $\mathfrak{G}$ to the coefficient group J; this process induces the purely algebraic description of B_1, as the group of functions f from $\mathfrak{G}$ to J, which satisfy (1) $h\in\mathfrak{H}$ and $a\in\mathfrak{G}\Rightarrow f(ha)=f(h)+f(a)$; (2) for each $a\in\mathfrak{G}$, $f(xa)=f(x)$ for all $x\in\mathfrak{G}$ except those in a finite number of right cosets of $\mathfrak{H}$, modulo the subgroup of functions which vanish on $\mathfrak{H}$. The construction is related to that of Eckmann [Proc. Nat. Acad. Sci. U. S. A. **33**, 275–281, 372–376 (1947); these Rev. **9**, 244, 298]. If a $\bar{\bar{K}}$ covers $\bar{K}$, the map of $B_1(\bar{\bar{K}})$ into $B_1(\bar{K})$ is defined; it can be described algebraically. The subgroup E_1 of B_1, the annihilator of the 1-cycles, is also described algebraically, together with B_1/E_1. It is shown for any countable complex that B_1 (integral coefficients) is a free Abelian group; this is proved separately for E_1 and B_1/E_1, by different arguments. The rank of E_1 determines the numbers of end points of the complex. Applications to homotopy properties of 3-manifolds are given, for example the following. For a finite M^3 without boundary, π_2 is determined by π_1; it is free to rank 0, 1 or ∞, depending on the number of endpoints of the universal covering. If M^3 has a boundary, but no 2-sphere in the boundary, and π_1 has 1 or 2 ends, then M^3 is aspherical. The well-known conjecture about asphericity of knots is equivalent to the conjecture that all knot groups have 1 or 2 ends. If g is the Abelian fundamental group of a finite orientable M^3, then g is cyclic, or free of rank two or three. The theorem of Fox, that Abelian fundamental groups of polyhedra in 3-space are free of rank 0, 1 or 2, follows from one of the results. *H. Samelson* (Ann Arbor, Mich.).

Postnikov, M. M. Homology invariants of continuous mappings. Doklady Akad. Nauk SSSR (N.S.) **66**, 161–164 (1949). (Russian)

The theory of obstructions to extensions and homotopies of continuous mappings for the case of mappings of complexes into a space Y which is assumed to be simple in all dimensions has been developed by several authors [e.g., S. Eilenberg, Ann. of Math. (2) **41**, 231–251 (1940); these Rev. **1**, 222]. This note gives a brief outline of how such a theory could be developed in case the assumption that Y is simple is removed. It is then necessary to use cohomology groups with local coefficients in the sense of Steenrod [Ann. of Math. (2) **44**, 610–627 (1943); these Rev. **5**, 104]. Moreover, in this case the fundamental group may be non-Abelian, and hence cannot be used as a coefficient group for cohomology. To get around this difficulty, the author introduces 1-dimensional cochains with coefficients in a non-Abelian group as was done by H. Robbins [Trans. Amer. Math. Soc. **49**, 308–324 (1941), in particular, pp. 323–324; these Rev. **3**, 141]. The author does not consider the question as to whether the 1-dimensional cohomology group thus defined is a topological invariant of a complex or not. [Reviewers' note: obstructions with coefficients in a local system of groups have previously been considered by A. Komatu [Jap. J. Math. **17**, 201–228 (1941); these Rev. **7**, 470].] *R. H. Fox and W. S. Massey* (Princeton, N. J.).

Postnikov, M. M. Classification of the continuous mappings of an arbitrary n-dimensional polyhedron into a connected topological space which is aspherical in dimensions greater than unity and less than n. Doklady Akad. Nauk SSSR (N.S.) **67**, 427–430 (1949). (Russian)

Let K be an n-dimensional complex, and Y an arcwise connected topological space whose homotopy groups $\pi_q(Y)$ are trivial for $1<q<n$. Using the methods outlined in a previous note [see the preceding review], the author gives necessary and sufficient conditions that two maps $f, g: K\to Y$ be homotopic. Essential use is made of the invariant cohomology class $\mathbf{k}^{n+1}\in H^{n+1}[\pi_1(Y), \pi_n(Y)]$ introduced by Eilenberg and MacLane [Proc. Nat. Acad. Sci. U. S. A. **32**, 277–280 (1946); Trans. Amer. Math. Soc. **65**, 49–99 (1949); these Rev. **8**, 398; **11**, 379]. No proofs are given. [Reviewers' note: this homotopy classification problem has also been solved by P. Olum, see Bull. Amer. Math. Soc. **53**, 1132 (1947). The case in which the group $\pi_1(Y)$ acts trivially on $\pi_n(Y)$ has been treated by Hu [see the following review].] *R. H. Fox and W. S. Massey* (Princeton, N. J.).

Hu, Sze-tsen. On homotopy classification of mappings. Acad. Sinica Science Record **2**, 227–244 (1949).

The author proves a homotopy-classification theorem for maps of an n-complex X into an n-simple space Y with $\pi_r(Y)=0$ for $1<r<n$; this includes the case $1\leqq r<n$ [Hopf] and $1<r$ [Hurewicz]. Definition: two maps $f, g: X\to Y$ are r-homotopic, if $f|X^r\simeq g|X^r$. An $f: X\to Y$ determines a homomorphism (-class) $c_f:\pi_1(X)\to\pi_1(Y)$ (f-images of loops with a base point). It is shown, as first part of the classification, that f and g are $(n-1)$-homotopic, if $c_f=c_g$. [In the proof the author uses a different homomorphism $k_f:\pi_1(X)\to\pi_1(Y)$: assume $f(X^0)=y_0$; each edge v_iv_j determines, as usual, a generator g_{ij} of $\pi_1(X)$; the f-image of v_iv_j is a closed curve, and $k_f(g_{ij})$ is the corresponding element of $\pi_1(Y)$. It is shown that the class of k_f is the c_f above.] Let μ be the $(n-1)$-homotopy class of f; each $\xi\in\pi_1(Y)$ determines an element $\gamma_\mu(\xi)$ of the cohomology group $H^n(X, \pi_n(Y))$, as an obstruction to extending to $X\times I$ a map, given on $X\times 0$

and $X\times 1$ as just f, and on vertex$\times I$ derived from ξ in a certain way. Actually γ_p is a homomorphism; the image group is called $R_p^n(X, \pi_n(Y))$. If g is another member of μ (i.e., $c_f = c_g$), one can deform g to g' so that $f|X^{n-1} = g'|X^{n-1}$; the difference cocycle $d^n(f, g')\epsilon H^n(X, \pi_n(Y))$ is then defined, and one finds, considering the choices made at various stages, that the residue class of d^n mod R_p^n depends only on the homotopy classes of f and g, and one gets the classification theorem saying that the homotopy classes contained in μ are in one-to-one correspondence with the elements of the factor group $Q_p^n = H^n/R_p^n$. *H. Samelson.*

Hu, Sze-tsen. A cohomology theory with higher coboundary operators. I. (Construction of the groups.) Nederl. Akad. Wetensch., Proc. **52**, 1144–1150 = Indagationes Math. **11**, 418–424 (1949).

A cohomology theory in topological spaces is given in terms of Wallace cochains [cf. Spanier, Ann. of Math. (2) **49**, 407–427 (1948); these Rev. **9**, 523]. For any positive integer p and integers $0\leqq i_1 < i_2 < \cdots < i_p \leqq m+p$ define the projection $\prod_{i_1\cdots i_p}(x_0, \cdots, x_{m+p}) = (x_{j_0}, \cdots, x_{j_m})$, where $i_1, \cdots, i_p, j_0, \cdots, j_m$ is a permutation of the first $m+p+1$ integers, and $j_0 < j_1 < \cdots < j_m$. For any m-cochain f^m the p-coboundary is the $(m+p)$-cochain

$$\delta^p f^m(x_0, \cdots, x_{m+p}) = \sum \pm \begin{pmatrix} 0, \cdots, p-1; p, \cdots, m+p \\ i_1, \cdots, i_p; j_0, \cdots, j_m \end{pmatrix} \times f^m(\textstyle\prod_{i_1, \cdots, i_p}(x_0, \cdots, x_{m+p})),$$

where the summation is extended over all choices of integers $0\leqq i_1 < i_2 < \cdots < i_p \leqq m+p$, where $i_1, \cdots, i_p, j_0, \cdots, j_m$ is the set of the first $m+p+1$ integers, where $j_0 < j_1 < \cdots < j_m$, and where the $\pm$ is taken according as the indicated permutation is even or odd. It is shown that $\delta^p\delta^q f^m = \delta^q\delta^p f^m = \theta(p, q)\delta^{p+q} f^m$, where $\theta(p, q) = 0$ if pq is odd, and

$$\theta(p, q) = \{[p/2] + [q/2]\}!/[p/2]![q/2]!$$

otherwise ($[x]$ is the largest integer not exceeding x). Let p, q be two integers, G an Abelian group with $\theta(p, q)g = 0$ for each $g\epsilon G$. Using δ^p one defines the group of m-cocycles $Z^m_{(p)}$ as usual, and with δ^q the group of m-coboundaries $B^m_{(q)}$; the assumption on G gives $B^m_{(q)} \subset Z^m_{(p)}$ and $H^m_{(p,q)} = Z^m_{(p)}/B^m_{(q)}$ is the (p, q) m-cohomology group of the space over G.

J. Dugundji (Los Angeles, Calif.).

Hu, Sze-Tsen. A cohomology theory with higher coboundary operators. II. Verification of the axioms of Eilenberg-Steenrod. Nederl. Akad. Wetensch., Proc. **53**, 70–76 = Indagationes Math. **12**, 1–7 (1950).

The author continues the study of the cohomology theory he initiated [see the preceding review] by showing that the Eilenberg-Steenrod axioms 1–3 and 5–6 hold, but the dimension axiom 7 does not. It is not determined whether the homotopy axiom 4 holds. *J. Dugundji.*

Hu, Sze-tsen. On the Whitehead group of automorphisms of the relative homotopy groups. Portugaliae Math. **7** (1948), 181–206 (1950).

An alternative formulation of a group introduced by J. H. C. Whitehead [Ann. of Math. (2) **49**, 610–640 (1948); these Rev. **10**, 392] is given, which is somewhat more general, and more systematic, than the original. Let $Y_0 \subset Y$, where Y_0 is closed in Y, be both ANR, and $y_0\epsilon Y_0$. Let Ω be the class of all maps of the tube $Y_0\times I$ into Y satisfying (a) $f|Y_0\times 0 = \text{identity} = i$, (b) $f(Y_0\times 1)\subset Y_0$, $f(y_0\times 1) = y_0$ and (c) for $f|Y_0\times 1$ there exists a $\varphi: Y_0\to Y_0$ with $\varphi f \simeq f\varphi \simeq i$ in Y_0 relative to y_0. Two elements of Ω are equivalent if they can be deformed to one another keeping the ends of the tube in Y_0 and the image of $y_0\times 1$ at y_0 during the entire deformation. Two elements $f, g\epsilon\Omega$ can be "multiplied" by the rule

$$f\cdot g(y, t) = \begin{cases} f(y, 2t), & 0\leqq t\leqq \frac{1}{2}; \\ g(f(y, 1), 2t-1), & \frac{1}{2}\leqq t\leqq 1 \end{cases}$$

and $f\cdot g\epsilon\Omega$ also. With these operations, the equivalence classes in Ω form a group $W(Y, Y_0, y_0)$, the Whitehead group of the pair (Y, Y_0) at y_0. The groups $W(Y, Y_0, y_0)$ are a system of local groups in Y_0, in the sense of Steenrod.

The structure of W is next investigated. There is a homomorphism $\kappa: W\to\pi_1(Y, y_0)$ induced by tracing the image of y_0 in the tube representing $w\epsilon W$, and a homomorphism $\lambda: \pi_1(Y_0, y_0)\to W$ by extending an obvious map of $Y_0\times 0 \cup y_0\times I$ over the tube; the image of λ is a normal subgroup. It follows easily that $\kappa\lambda$ is the natural homomorphism $j: \pi_1(Y_0, y_0)\to\pi_1(Y, y_0)$. Various relations follow from this. Example: j an isomorphism on implies $W = \text{kernel } \kappa \otimes \text{image } \lambda$ (direct product). [Reviewer's remark: the techniques used in this section 5 are needlessly complicated, all results being simple algebraic consequences of the relation $j = \kappa\lambda$.]

The operation of W on the relative homotopy groups originally given by Whitehead is extended by the author to give W as a group of (right) operators on the entire homotopy sequence (i.e., the obvious commutativity relations hold), and, in fact, for $\alpha\epsilon\pi_n(Y, y_0)$, $\alpha\epsilon\pi_1(Y, y_0)$ and a_0, α_0 elements in the same groups for Y_0, then $aw = a\kappa(w)$, $a_0\alpha_0 = a_0\lambda(\alpha_0)$ express the relation of these operations to the classical operations. *J. Dugundji* (Los Angeles, Calif.).

Hu, Sze-Tsen. Boundedness in a topological space. J. Math. Pures Appl. (9) **28**, 287–320 (1949).

A "boundedness" is an ideal B of sets in a topological space X: its members are called "bounded." Let B^o (B^-) be the ideal generated by the interiors (closures) of the bounded sets. Then B is called "open" if $B^o = B$, "closed" if $B^- = B$, "proper" if $B^- = B^o$, and X is locally bounded if the open bounded sets cover X. The paper first develops these notions extensively. The object "universe" (= space plus boundedness) is not a topological invariant. Homeomorphisms between universes preserving boundedness are called "complete." It is shown that the homeomorphisms involved in Urysohn's metrization-imbedding theorem and in the Menger-Nöbeling imbedding can be chosen so as to be complete, the bounded sets in the embedding space being the usual ones. Finally, observing that the compact sets form a boundedness, it is observed that in Alexandroff's definition of Čech homology [P. Alexandroff, Trans. Amer. Math. Soc. **49**, 41–105 (1941), p. 82; these Rev. **2**, 323] the compact sets may be replaced by the sets of an arbitrary boundedness. The duality theorem can be extended. For normal, locally bounded proper universes, there results a generalization of Alexandroff's form of Kolmogoroff's duality theorem [op. cit., p. 86]. *R. Arens.*

Shirai, Tameharu. On the relations between the set and its distances. Mem. Coll. Sci. Kyoto Imp. Univ. Ser. A. **22**, 369–375 (1939).

Let

$$f_1(x, y) = |x-y|, \quad f_2(x, y) = \max(|x|, |y|), \quad f_3(x, y) = |x| + |y|,$$

when x, y are real and unequal, and let $f_i(x, x) = 0$ $(i = 1, 2, 3)$. Let $f(x, y) = \phi(f_i(x, y))$, where ϕ is a real-valued function.

The author gives geometrical conditions, expressed in terms of sections of the surface $z=f(x, y)$, for f to satisfy the triangle inequality. *A. H. Stone* (Manchester).

Koseki, Ken-iti. Two theorems on the connected sets. Mem. Coll. Sci. Kyoto Imp. Univ. Ser. A. 24, 149–158 (1944).

Let M be a locally compact connected metric space; a set $E\subset M$ is a "division set of order k" if E is connected and $M-E$ has k components, where $k\geqq 2$. The author proves the following theorems. (I) If M has a largest division set, M is a simple arc. (II) If (i) M has a division set, (ii) every division set is of order 2, (iii) the intersection A of all division sets is nonempty, and (iv) A is closed, then M is homeomorphic to a θ or 8. [Condition (i) was omitted, and (iv) is superfluous. The proof of (II) could have been greatly shortened by using known properties of cut-points.] *A. H. Stone* (Manchester).

Newman, M. H. A. Local connection in locally compact spaces. Proc. Amer. Math. Soc. 1, 44–53 (1950).

The author proves that a locally compact metric LC^1 lc^n space X is LC^n ($n\geqq 2$), where the local connectednesses are either all uniform or all relative (i.e., the theorem holds for any open $G\subset X$ which is LC^1 lc^n relative to X). This represents an essential generalization of a theorem due to Hurewicz [Fund. Math. 25, 467–485 (1935)] for the compact case. The method used follows that of Hurewicz, with the crucial passage from ϵ-homotopy to true homotopy being possible because of the following equivalent formulation of the LC^n condition in locally compact metric spaces: X is LC^n if and only if there exists a $\zeta(x, \delta)$ such that, given any compact $F\subset U(x, \zeta(x, \delta))$ there is a compact $F'\subset U(x, \delta)$ with the property that every $f: S^p\rightarrow F$, $0\leqq p\leqq n$, is contractible in F'. (A similar property holds for lc^n.) *J. Dugundji* (Los Angeles, Calif.).

Denjoy, Arnaud. La multiconnexité des ensembles. C. R. Acad. Sci. Paris 230, 693–695 (1950).

By a recursion procedure the author extends the Cantor notion of connectedness (well-chainedness) to a higher dimensional connectedness. For example, a closed set M is biconnected provided that for any two distinct closed connected subsets Γ_1 and Γ_2 of it whose intersection contains points A and B, one can extend ϵ-chains joining A and B in Γ_1 and Γ_2 to a 2-dimensional ϵ-lattice in M for each $\epsilon>0$; and similarly triconnectedness is defined with respect to pairs of biconnected subsets and so on. It is shown that every Cartesian space of any dimensionality is pluriconnected in all orders. Also, pluriconnectedness in a given order s relative to a pair of $(s-1)$-connected subsets is an inductive property so that irreducibly pluriconnected subsets can be obtained in compact separable spaces. The author's ideas were indicated in communications published in the same journal in 1911 and, of course, they closely resemble notions leading to the fundamental group and related groups which are well known and much in use today. *G. T. Whyburn* (Charlottesville, Va.).

Denjoy, Arnaud. Les espaces biconnexes. C. R. Acad. Sci. Paris 230, 797–800 (1950).

Using the notion of "biconnectedness" referred to in the preceding review, the author proves that if two disjoint regions in a compact metric and biconnected space have a pair of boundary points in common, each of their boundaries is connected (i.e., uniconnected) between these two points. It is pointed out that this conclusion does not hold on the torus which is not biconnected; and a conjecture is made concerning an extension to higher dimensionally connected sets in the author's sense. *G. T. Whyburn.*

Anderson, R. D. Concerning upper semi-continuous collections of continua. Trans. Amer. Math. Soc. 67, 451–460 (1949).

It was proved independently by Hurewicz [Fund. Math. 15, 57–60 (1930)] and Mazurkiewicz [C. R. Congrès Math. des Pays Slaves, Warsaw, 1929, pp. 66–71 (1930)] that if M is any compact metric continuum then there exists in Euclidean 3-space a one-dimensional continuum K and a monotone mapping $f(K)=M$. In the present paper it is shown that, with the additional hypothesis that M is a continuous curve, the continuum K can also be taken to be a continuous curve. A similar result holds if M is not compact. In this case K must be unbounded, but each set $f^{-1}(y)$, $y\epsilon M$, is bounded. Question by the reviewer: Can K and f be found so that, in addition, for each $y\epsilon M$ the set $f^{-1}(y)$ is a continuous curve? *J. H. Roberts.*

Wallace, A. D. Cyclic invariance under multi-valued maps. Bull. Amer. Math. Soc. 55, 820–824 (1949).

Let X, Y be compact connected Hausdorff spaces. A multi-valued map f assigning a subset $f(x)$ of Y to each point of X is continuous provided that the image of a closed subset of X and the inverse of a closed subset of Y are both closed. The continuous map f is anarthric if for y in Y no point in $X-f^{-1}(y)$ separates $f^{-1}(y)$ in X. A number of theorems, many generalizing classical results on nonalternating transformations and on mappings of A-sets [Wallace, Duke Math. J. 9, 487–506 (1942); these Rev. 4, 147] are proved. For example, if f is anarthric and the image of a cut point is a point, then the image of an A-set is an A-set or a point. *G. S. Young* (Ann Arbor, Mich.).

Floyd, E. E. A nonhomogeneous minimal set. Bull. Amer. Math. Soc. 55, 957–960 (1949).

A space X is minimal with respect to a transformation T of X onto X if no proper closed subset of X is mapped into itself by T. It is shown that there exist a compact plane set X, of dimension 0 at some points and dimension 1 at others, and a homeomorphism T of X onto X such that X is minimal with respect to T. *G. S. Young.*

GEOMETRY

★Bouligand, Georges. Les Principes de l'Analyse Géométrique. Tome II. (A) Opérations et Groupes, Topologies. (B) Géométrie Infinitésimale Directe. Fascicule (A): Base Méthodologique. Librairie Vuibert, Paris, 1950. xxii+209 pp. 1100 francs.

We have vol. I

[Cf. tome I, 3d ed., 1949; these Rev. 10, 568.] This book is a survey of a large area of modern mathematics, little of which is usually called "geometry," but which the author wishes to have available for use in Part (B) of this work which is to be titled "Géométrie Infinitésimale Directe." The material is presented in a clear, straightforward fashion with numerous specific examples. References are made to the literature, largely that published in France. The book thus serves as a quick introduction to the fields covered,

but the serious reader will usually wish more information than is presented here. The following are the chief topics included: sets, abstract groups, applications of groups, fields, general topology, combinatorial topology, measure theory, lattices, continuity and related notions, connectivity, the continuum, the notion of a curve, sequences of sets with applications, and various topics in the philosophy of mathematics. *C. B. Allendoerfer* (Haverford, Pa.).

✶Duporcq, Ernest. Premiers principes de géométrie moderne. Troisième édition par Raoul Bricard. Gauthier-Villars, Paris, 1949. i+174 pp. 375 francs.

"Nouveau tirage" of the edition of 1938.

Kuiper, N. H. A closure theorem. Simon Stevin **27**, 6–15 (1949). (Dutch)

In the plane the points O_i $(i=1, \cdots, r)$ and the angle α are given. The rotation (O_1, α) moves the point P_0 into P_1, (O_2, α) moves P_1 into P_2 and so on. If $n\alpha \not\equiv 0 \pmod{2\pi}$ then there is one point P_0 so that $P_n=P_0$; if $n\alpha\equiv 0$ and for one point P_0 the relation $P_n\equiv P_0$ holds, then for an arbitrary point Q_0 we have $Q_n=Q_0$. Generalizations. Applications to several problems of elementary geometry. *O. Bottema.*

Beth, H. J. E. The deformable quadrangle. Nieuw Tijdschr. Wiskunde **37**, 161–165 (1949). (Dutch)

The author considers the quadrangles $ABCD$, the sides of which are given, A and B being fixed. If ABC_0D_0 is one of the quadrangles of the system whose points are on a circle and P the centre of the rotation which moves C_0D_0 into CD, then P is regarded as the image of $ABCD$. The locus of P is an elliptic circular cubic γ. There are pairs of quadrangles of the system so that the angles (AB, CD) and (BC, DA) have the same values. The images of these pairs are the pairs of an involution on γ. The representation of the system on a cubic curve, given by the author, is different from that of Darboux and of Weisz. *O. Bottema* (Delft).

Boggio, Tommaso. Il calcolo geometrico di Peano. Univ. e Politecnico Torino. Rend. Sem. Mat. **8**, 71–92 (1949).

Let ABC be three points in a fixed plane. Denote by ABC the signed area of the triangle ABC. The symbol ABC is called a "tripoint." The equation $ABP=ABC$ is the equation, for variable P, of the straight line through C parallel to AB, and $ABP=0$ is the equation of the line AB. The symbol AB is called a "bipoint"; it is an operation on AB such that if C is any point whatever then $(AB)C=ABC$, and $AA=0$. Let A_i, $i=1, 2, \cdots, r$, be r points in the plane and let m_i be r real numbers. The expression $\sum^r_{i=1}M_iA_i$ is defined as such a point that for every pair PQ, $(\sum^r_{i=1}M_iA_i)PQ=\sum^r_{i=1}M_i(A_iPQ)$. Using these formalisms and others, and extensions of them to points in a space of three dimensions, applications are made to familiar locus problems giving rise to straight lines, conics, cubic curves, planes, quadrics, cubic and quartic surfaces.

V. G. Grove (East Lansing, Mich.).

Lauffer, R. Der Satz von Ptolemaios. Österreich. Akad. Wiss. Math.-Nat. Kl. S.-B. IIa. **157**, 53–61 (1949).

Verf. gibt den Satz in folgender Fassung: Liegen A, B, C, D in dieser Reihenfolge auf einem Kreis, dann ist die Summe der Rechtecke, gebildet aus den Strecken AB und CD, bzw. BC und DA, inhaltsgleich dem Rechteck aus den Strecken AC und BD, und es gelingt ihm so den Beweis lediglich mit den Axiomen der Verknüpfung, Anordnung, Kongruenz und dem euklidischen Parallelenaxiom, also ohne Voraussetzung der Stetigkeit (und der Ähnlichkeitslehre) zu beweisen. Er gibt zwei Beweise: (a) mit dem Desargueschen Satz, (b) mit raumgeometrischen Betrachtungen. Erweiterung und Umkehrung des Satzes. *O. Bottema* (Delft).

Bagchi, Haridas. Note on critic centres and coincidence points of cubic curves. J. Indian Math. Soc. (N.S.) **13**, 73–75 (1949).

If Δ_i $(i=0, \cdots, 3)$ are the four triangles each of which contains the nine inflexions of a plane cubic C, the four pairs of cubic curves which meet C in its 72 coincidence points are all equianharmonic, and those of any pair have one of the Δ_i as common self-polar triangle and the other three as common Hart triangles. *J. G. Semple* (London).

✶Gibson, Robert Wilder. Projective Geometry with Coordinates from a Commutative Primary Ring. Abstract of a Thesis, University of Illinois, 1943. ii+4 pp.

Here the projective geometry is the lattice of admissible subgroups, in an Abelian group with multiplicative operators from a commutative primary ring R. The dimension of a subgroup is defined as one less than the number of elements in a basis. Since R may contain zero divisors a line may properly contain another line. Nevertheless certain geometrical concepts, in particular that of cross-ratio, retain their validity. Using cross-ratios the author asserts he can show that projectivities are induced by linear transformations.

M. Hall (Columbus, Ohio).

✶Gingerich, Hugh Francis. Generalized Fields and Desargues Configurations. Abstract of a Thesis, University of Illinois, 1945. 12 pp.

Designate one line of a projective plane as the line h at infinity and another as the line $y=x$, through a point called the origin (0, 0). Then addition may be defined in terms of lines $y=x+b$ parallel to $y=x$, and multiplication in terms of lines $y=xm$ through the origin. In this abstract a study is made of the precise relationship of the properties of the operations to the validity of restricted forms of the theorem of Desargues. For example, there exist collineations leaving the points of h fixed, transitive on the lines of slope zero, if and only if lines satisfy equations $y=(x+b)a$ and addition is associative. *M. Hall* (Columbus, Ohio).

Bruins, E. M. On the symbolical method. II. Nederl. Akad. Wetensch., Proc. **52**, 3–13 = Indagationes Math. **11**, 35–45 (1949).

In the first paper of the series [same Proc. **51**, 1270–1276 = Indagationes Math. **10**, 414–420 (1948); these Rev. **10**, 469] the 3-symbol bracket factors in the symbolic notation for ternary forms representing plane curves were factorized into 2-symbol brackets by the adjunction of a normal curve of degree 2. The method is here further employed to obtain the Veronese-properties of the Pascal hexagons of 6 points on a conic. By the method employed the ternary geometry of the conic is reduced to a binary geometry. Then, the Pascal theorem being proved, the Veronese-properties are the simplest theorems one can write down. The theorems proved concern such properties as the concurrence of the Pascal lines at Pascal points, Steiner points and Kirkman points, the perspectives of the 15 Veronese triangles in 20 triples with Steiner points as centres, the collinearity of the Steiner points in sets of 4 on the Steiner-Plücker lines and in sets of 3 on the Bayley-Salmon lines, and further relations of this kind. *D. E. Littlewood* (Bangor).

Bruins, E. M. On the symbolical method. III. Nederl. Akad. Wetensch., Proc. **52**, 705–713 = Indagationes Math. **11**, 235–243 (1949).

This paper is a continuation of two earlier papers [cf. the preceding review] and uses the same methods. Pascal's "hexagramma mysticum" is discussed. Various conditions are obtained for the degeneration of the hexagon. Thus, e.g., if two Pascal points coincide an invariant of the sextic is zero. Of the 15 possible relations of this kind, if two are satisfied simultaneously and these involve a common line, then the hexagon is composed of a tetragon and one of its diagonals. If two are satisfied not involving a common line, then a third is automatically satisfied and the hexagon is composed of two triples having the same hexagon. The possible forms of degeneracy are exhaustively discussed.

D. E. Littlewood (Bangor).

Primrose, E. J. F. Projective mutual invariants of a space cubic and a linear complex. Proc. Cambridge Philos. Soc. **46**, 195–198 (1950).

There are 3 simultaneous invariants of a twisted cubic K and a linear complex L. The first invariant C is zero if the cubic becomes a plane curve. The others are obtained by projecting the cubic into a conic S, where the chords of the conic which belong to the complex are projected into lines touching another conic Σ'. The simultaneous invariants of S and Σ' are simultaneous invariants of the two forms. If the cubic is put parametrically into the form $(1, t, t^2, t^3)$ and the complex is $\sum C_{ij}P_{ij}=0$ $(i, j=0, 1, 2, 3)$, the independent invariants are

$$M=C_{12}-3C_{03},\ N=\begin{vmatrix} 2C_{23}\ , & C_{31}, & C_{12}-C_{03} \\ C_{31}\ , & 2C_{03}, & -C_{02} \\ C_{12}-C_{03}, & -C_{02}, & 2C_{01} \end{vmatrix}.$$

Some interpretations of the invariants are made, e.g., as follows. If $M=0$ the linear complex L' associated with K and L is apolar. If $N=0$ the conic Σ' regarded as an envelope degenerates. The complexes L and L' determine a linear congruence the directrices of which coincide if $M^2+12C=0$. Analogous results are obtained in the 5-space of the line coordinates.

D. E. Littlewood (Bangor).

Bessemoulin, J., et Pône, R. Détermination des routes aériennes à durée minimum. J. Sci. Météorologie **1**, 101–121 (1949).

Convex Domains, Extremal Problems

Aczél, John, and Fuchs, Ladislas. A minimum-problem on areas of inscribed and circumscribed polygons of a circle. Compositio Math. **8**, 61–67 (1950).

Let P_1 be a polygon inscribed in the unit circle and containing the origin, and let P_2 be the circumscribing polygon whose points of contact with the unit circle are the vertices of P_1. Let A_1 and A_2 be the areas of P_1 and P_2, respectively. P. Szász conjectured that the minimum for A_1+A_2 occurs when P_1 is a square. The authors prove this conjecture.

A. W. Goodman (Lexington, Ky.).

Fejes Tóth, László. Über dichteste Kreislagerung und dünnste Kreisüberdeckung. Comment. Math. Helv. **23**, 342–349 (1949).

Let T be a convex 2-dimensional region of area T. Suppose that a circles of radius r and total area $s=\pi r^2 a$ may be placed in T without overlapping. Then it is shewn that (1) $s<\sqrt{3}\pi T/6$ if $a\geqq 2$. Also, if T can be covered completely by A (overlapping) circles of radius r and total area $S=\pi r^2 A$, then (2) $S>2\sqrt{3}\pi T/9$ if $A\geqq 2$. Analogous but sharper results are proved when T is the surface of a sphere, namely, for $n\geqq 3$ circles,

(3) $s\leqq \frac{1}{2}n\{1-\frac{1}{2}\csc(\frac{1}{6}n\pi/(n-2))\}T<\sqrt{3}\pi T/6,$

(4) $S\geqq \frac{1}{2}n\{1-3^{-\frac{1}{2}}\cot(\frac{1}{6}n\pi/(n-2))\}T>2\sqrt{3}\pi T/9.$

Both (3) and (4) emerge as special cases of a single more general theorem which connects the packing and covering constants for any arrangement of circles (overlapping or not) on the unit sphere. The proof of this theorem, which cannot be stated in a few lines, makes use of an inequality for integrals obtained by the author in an earlier paper [Amer. J. Math. **70**, 174–180 (1948); these Rev. **9**, 460].

R. A. Rankin (Cambridge, England).

Fejes Tóth, L. On the densest packing of circles in a convex domain. Norske Vid. Selsk. Forh., Trondhjem **21**, no. 17, 68–71 (1949).

Let a convex domain T be decomposed into n convex domains $(n\geqq 2)$ of areas $t_1, t_2, \cdots, t_n$ and of perimeters $l_1, l_2, \cdots, l_n$. It is shown that $\sum_{i=1}^{n} l_i^2/t_i\geqq 8n\sqrt{3}$. The result is used to show that if n nonoverlapping circles of unit radius are contained in a convex domain T, then the area of T exceeds $2n\sqrt{3}$. This latter result is a refinement of a result due to A. Thue [Forhandlingerne ved de Skandinaviske Naturforskeres, v. 14, Copenhagen, 1892, pp. 352–353].

C. A. Rogers (Princeton, N. J.).

Verblunsky, S. On the least number of unit circles which can cover a square. J. London Math. Soc. **24**, 164–170 (1949).

As an improvement of a result of R. Kershner [Amer. J. Math. **61**, 665–671 (1939); these Rev. **1**, 8] concerning the least number $N(\sigma)$ of unit circles which can cover a square of side σ the author proves that for all sufficiently large σ: $3^{\frac{3}{2}}N(\sigma)>2\sigma^2+\sigma$. This inequality is contained in a more general result of the reviewer [see the two preceding reviews] which gives, for instance, $3^{\frac{3}{2}}N(\sigma)>2\sigma^2+1.1\sigma-3$ for all $\sigma>0$.

L. Fejes Tóth (Budapest).

Inzinger, Rudolf. Über eine lineare Transformation in den Mengen der konvexen und der stützbaren Bereiche einer Ebene. Monatsh. Math. **53**, 227–250 (1949).

A transformation T operating on the convex domain C in the plane bounded by closed (convex) curves with continuous curvature is constructed with the following properties: T transforms a linear family of C's again into a linear family. Homothetic C's go into homothetic C's. Each circle is invariant. If C is not a circle, then TC has greater area than C, and the boundary of TC has the same length as that of C. The sequence T^nC tends uniformly to a circle. If C is not a circle, then TC has a larger inscribed circle than C, a smaller circumscribed circle, a smaller diameter, a larger thickness, the boundary of TC has a smaller maximal and a larger minimal curvature than that of C.

H. Busemann (Los Angeles, Calif.).

Berwald, L. Obere Schranken für das isoperimetrische Defizit bei Eilinien und die entsprechenden Grössen bei Eiflächen. Monatsh. Math. **53**, 202–210 (1949).

Let C_1, C_2 be closed convex curves in the plane with continuous nonvanishing curvatures. Let L_i and F_i denote length and area of C_i. If r is the largest number such that

rC_2 can roll in C_1 and R the smallest number such that C_1 can roll in RC_2, then $L_1^2L_2^2-16\pi^2F_1F_2\leqq 4\pi^2F_2^2(R-r)^2$ and the equality sign holds only if both C_1 and C_2 are circles. If C_2 is the unit circle then R and r are the maximal and minimal radii of curvature of C_1 so that the above relation becomes Bottema's inequality $L_1^2-4\pi F_1\leqq\pi^2(R-r)^2$. Let F_1, F_2 be closed convex surfaces in E_3 with continuous positive Gauss curvature. Denote by M_i, A_i, V_i the integral over the mean curvature, the area, the volume of F_i. If r is the largest number such that rF_2 can roll in F_1, and R the smallest number for which F_1 can roll in RF_2, then

$$M_1^2M_2^2-16\pi^2A_1A_2\leqq 4\pi A_2^2(R-r)^2,$$
$$M_1^2A_2^2-12\pi A_1M_2V_2\leqq 36\pi^2V_2^2(R-r)^2,$$
$$M_2^2A_1^2-12\pi A_2M_1V_1\leqq 4^{-1}M_1^2A_2^2(R-r)^2.$$

The equality sign holds in any one inequality only if both F_1 and F_2 are spheres. It is shown that the inequalities obtained in the special case where F_2 is the unit sphere can be improved. *H. Busemann* (Los Angeles, Calif.).

Santaló, L. A. An affine invariant for closed convex plane curves. Math. Notae 8, 103–111 (1948). (Spanish)

If lines in the plane are represented in normal form $x\cos\varphi+y\sin\varphi-h=0$ then $h^{-3}dhd\varphi$ is, except for a constant factor, the only density for lines not through the origin which is invariant under the unimodular affine transformation leaving the origin fixed. Let K be a convex curve with the origin p in its interior; then the measure of all lines exterior to K with respect to p as origin is $I(p)=\frac{1}{2}\int_0^{2\pi}h^{-2}d\varphi$. When p tends to K, $I(p)$ tends to ∞. The minimum of $I(p)$ is attained at exactly the point p_K, which is affinely connected with K and coincides with the center of K if K has a center. If F is the area and L_a the affine length of K, then $I(p_K)F\leqq\pi^2$, $I_mL_a^3\leqq 8\pi^4$ and the equality sign holds only when K is an ellipse. *H. Busemann.*

Tietze, Heinrich. Über stabile und indifferente Ruhelagen eines homogenen Zylinders. S.-B. Math.-Nat. Kl. Bayer. Akad. Wiss. **1948**, 299–301 (1949).

In Elemente der Math. 3, 97–100 (1948) [these Rev. 10, 141] the author proved a theorem quoted in the cited review. It is now shown that the condition that no circular arcs occur is unnecessary. *W. Feller* (Ithaca, N. Y.).

Algebraic Geometry

★Lefschetz, S. L'analysis situs et la géométrie algébrique. Gauthier-Villars, Paris, 1950. vi+154 pp. 650 francs.

"Nouveau tirage" of a work first published in 1924.

Chow, Wei-Liang. Über die Lösbarkeit gewisser algebraischer Gleichungssysteme. Comment. Math. Helv. **23**, 76–79 (1949).

The following generalization of a theorem of W. Habicht [same Comment. 18, 154–175 (1946); these Rev. 8, 61] is proved: let V be an algebraic variety of dimension $n-1$ in the projective m-space over an algebraically closed field K; let f_i $(1\leqq i\leqq n)$ be forms of degree h and g_i $(1\leqq i\leqq n)$ forms of degree k (in $m+1$ variables, with coefficients in K); assume that $\sum_{i=1}^n f_i(\xi)g_i(\xi)=0$ for every ξ in V, and that we have not at the same time $h=k$ and n even; then either $f_1, \cdots, f_n$ or $g_1, \cdots, g_n$ have a common zero on V. (The case considered by Habicht was the one in which $m=n-1$, V is the whole space and $g_i=x_i$). For the proof, it is assumed that $g_1, \cdots, g_n$ do not have any common zero on V; $f_1, \cdots, f_n$ are then replaced by the "most general forms" $F_1, \cdots, F_n$ which give rise to the equality $\sum_{i=1}^n F_i(\xi)g_i(\xi)=0$ for every ξ in V, and it is proved that $F_1, \cdots, F_n$ have a common zero on V. In order to do this, it is sufficient to prove that one of the common zeros of $F_1, \cdots, F_{n-1}$ on V (they exist for reasons of dimensions) is not a zero of g_n. This is accomplished by constructing explicitly specializations $F_i^*(1\leqq i\leqq n)$ of the forms F_i such that $F_1^*, \cdots, F^*_{n-1}$ have a common isolated zero on V which is not a zero of g_n. *C. Chevalley* (New York, N. Y.).

Tanturri, Giuseppe. Sistemi lineari di C^n piane i cui punti base sono flessi di ordine $n-2$. Univ. e Politecnico Torino. Rend. Sem. Mat. 8, 241–251 (1949).

L'auteur commence par étudier les systèmes linéaires de courbes C^n ayant des points bases où les tangentes, à contact maximum, sont fixes. Si trois tels points A, B, C ne sont pas alignés et si les tangentes en B et C se coupent en A', la tangente en A forme avec AB, AC, et AA' un rapport anharmonique: exp $(2i\pi/n)$. L'auteur étudie ensuite le cas d'un point base à tangente variable en généralisant un résultat de Calvi [Giorn. Mat. Battaglini (3) 72, 71–75 (1934)]. La condition nécessaire et suffisante pour que les courbes d'un systéme linéaire aient en un point base simple P une inflexion d'ordre r est que soit fixe la tangente d'inflexion ou bien sa résiduelle par rapport au système polaire de P. Outre les systèmes dont les points inflexionnels de base sont à tangentes toutes fixes, on obtient les systèmes suivants (et leurs sous-systèmes): le système de dimension $n+2$ des C^n ayant un point base inflexionnel maximal à tangente variable; les réseaux de C^n ayant $m\leqq n$ points bases inflexionnels maximaux alignés, dont un seul est à tangente variable; le réseau des C^n $(n>3)$ ayant deux points bases inflexionnels maximaux à tangentes variables; le réseau de cubiques ayant trois points bases inflexionnels alignés à tangentes variables; le faisceau de quartiques avec trois points bases d'ondulation non colinéaires, à tangentes variables; le faisceau syzygétique de cubiques. *L. Gauthier* (Nancy).

Morgantini, Edmondo. Sui fasci di curve piane razionali. Rend. Sem. Mat. Univ. Padova 18, 203–227 (1949).

An outstanding problem, for algebraic curve congruences of index unity in S_3, is that of their classification into types which are distinct over Cremona transformations. In this paper the author, limiting himself to congruences whose curves lie in the planes of a pencil, shows that these are reducible either to congruences of conics or to line-stars. This result is equivalent to the theorem that any given pencil Σ of rational plane curves of order $n>2$ is reducible, by Cremona transformation of the plane, and without introducing new arithmetical irrationalities, to a pencil of conics or lines. The proof of this theorem is in two parts. By using Noether's result that a rational C^n, if n is odd, can be transformed birationally into a line without introducing irrationalities relative to C^n, the author first shows that the theorem holds if n is odd, or if n is even and Σ has an odd number of base points of equal odd multiplicity. The second and main part of the proof, dealing with the case when n is even and Σ has even numbers of base points of equal odd multiplicity, employs a birational representation of the plane of Σ on a surface F^m of S_3 with an $(m-2)$-fold line S, the conics C^2 in which F^m is met by planes through S being projectively related to the curves of Σ. By results due to

Morin [Rend. Sem. Mat. Univ. Padova 9, 123–139 (1938)], F^m can be so chosen that the $3m-4$ degenerate conics of the pencil (C^2) have distinct components not intersecting on S, and F^m has no singularities, other than ordinary double points, not on S. When the pencil (C^2) has degenerate members which are rationally separable, Σ has only 4 base points of odd multiplicity; in this case, with certain exceptions, a suitable plane representation of F^m provides a Cremona transformation (not introducing any arithmetic irrationalities) of Σ into a pencil of conics. In the exceptional cases, and when none of the degenerate C^2 is rationally separable, a rational determination of a unisecant of the conics C^2 leads to a reduction of Σ to a pencil of lines.

J. G. Semple (London).

Büke, Macit. Les surfaces d'ordre *n* de del Pezzo dans l'espace projectif à *n* dimensions. Rev. Fac. Sci. Univ. Istanbul (A) **14**, 143–164 (1949). (French. Turkish summary)

This paper, of which the last of three chapters is still to come, is an extension to Del Pezzo surfaces of orders $5, \cdots, 9$ of the author's previous detailed study of Segre quartic surfaces [same Rev. (A) 12, 164–189, 255–288 (1947); these Rev. 9, 56, 609]. Chapter I deals with quintic Del Pezzo surfaces. In the first section the author discusses in detail the configuration of lines on F^5, using for this purpose a very convenient plane representation (by projection from a tangent plane) in which the 10 lines of the surface are represented by the joins of five points in the plane. The various sets of skew lines (doublets, two types of triplet, and quadruplets) are all enumerated, as also the various double threes and double pentagons. The type of configuration formed by the lines is shown to have the same number of degrees of freedom as F^5 itself. In the second section, the author discusses the group of the ten lines, this being isomorphic with the group of substitutions of the five base points in the plane representation, and he relates many of the properties already obtained for the lines to abstract properties of the group. The third section of this chapter gives the classification of F^5 in the real domain. From the plane representation, or by using continuous variation through nodal F^5, it appears that there are three types of real F^5. Surfaces of these three types have 10, 4 and 2 real lines, and they are non-orientable surfaces of characteristics 3, 1 and -1, respectively. Chapter II deals, in much the same manner, with all the Del Pezzo surfaces of order $n>5$; and it contains some general observations on the relations between the groups of the lines, and the topological characteristics, of all the Del Pezzo surfaces. *J. G. Semple.*

Edge, W. L. The Kummer quartic and the tetrahedroids based on the Maschke forms. Proc. Cambridge Philos. Soc. **45**, 519–535 (1949).

If $\Delta_1, \cdots, \Delta_5$ are the forms $x^4+y^4+z^4+t^4$, $xyzt$, $y^2z^2+x^2t^2$, $z^2x^2+y^2t^2$, $x^2y^2+z^2t^2$, the equation

$$\Delta_1+2D\Delta_2+A\Delta_3+B\Delta_4+C\Delta_5=0$$

represents a linear ∞^4 system Λ of quartic surfaces which contains, as is well known, an ∞^3 system Λ_K of Kummer surfaces, represented in the parameter space $S_4(A, B, C, D)$ by the Segre cubic primal Γ whose equation is

$$ABC+D^2+4=A^2+B^2+C^2.$$

In part I of the present paper the author indicates the substantial advantages to be gained, in the study of the system Λ, by writing the equation of this system in the form $\sum_1^6 J_i\Phi_i=0$, $\sum_1^6 J_i=0$, where the J_i are parameters and the Φ_i are the six Maschke forms $\Delta_1-6(\Delta_3+\Delta_4+\Delta_5)$, $\Delta_1+6(-\Delta_3+\Delta_4+\Delta_5)$, $\Delta_1+6(\Delta_3-\Delta_4+\Delta_5)$, $\Delta_1+6(\Delta_3+\Delta_4-\Delta_5)$, $-2\Delta_1+24\Delta_2$, $-2\Delta_1-24\Delta_2$, these being connected by the identity $\sum\Phi_i=0$. The main points are that any two of the surfaces $\Phi_i=0$ bear the same relation to one another; the vanishing of the differences $\Phi_i-\Phi_j$ give the sets of four planes which form Klein's 15 fundamental tetrahedra; the sums of triads of the Φ_i give the squares of Klein's 10 fundamental quadrics; and the equations of Γ in the space of the parameters J_i are $\sum J_i^3=\sum J_i=0$.

Part II is mainly concerned with the construction of the discriminant D of Λ; this is obtained in the form $D\equiv(\sum J_i^3)^{16}\prod(J_i+J_j)^4$, each linear factor (associated with a solid which meets Γ in three planes) arising from a family of quartic surfaces of Λ which have nodes at the vertices of a fundamental tetrahedron. The systems of scrolls of Λ, each with a pair of opposite edges of one of the tetrahedra as double lines, are also exhibited.

Part III deals with the tetrahedroids belonging to Λ, a tetrahedroid being a Kummer surface whose 16 nodes lie by fours in the faces of a fundamental tetrahedron. For a surface of Λ to be a tetrahedroid it is necessary that two of the J_i should be equal. This gives principally 15 doubly infinite systems of tetrahedroids represented by the 4-nodal cubic surfaces in which Γ is met by the solids Π_{ij} whose equations are $J_i=J_j$. Multiple tetrahedroids, and, in particular, 30 sextuple tetrahedroids, correspond to points common to several of these cubic surfaces. It appears, however, that Λ contains 30 other sextuple tetrahedroids, not represented by points of Γ, which correspond to intersections of sets of four of the solids Π_{ij}. One of these has equation $x^4+y^4+z^4+t^4=4xyzt$. The paper concludes with a discussion of the simply infinite systems of triple tetrahedroids, these being represented on Γ by nodal cubic curves.

J. G. Semple (London).

Dedò, Modesto. Invarianti per trasformazioni puntuali singolari dei rami superlineari delle curve algebriche piane. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) 11(80) (1947), 53–58 (1949).

Consider the algebraic branch defined parametrically by

$$x=t^\nu,\quad y=\sum_{r=1}^{s} m_r t^{r\nu}+\sum_{i=1}^{\nu-1} a_i t^{s\nu+i}+t^{(s+1)\nu}q(t),$$

where $q(t)$ is a power series, $s>1$, $\nu>3$. Set

$$I=2(\nu+1)a_1a_3-(2\nu+3)a_2^2,$$
$$J=3(\nu+1)^2a_1^2a_4-6(\nu+1)(\nu+2)a_1a_2a_3+(\nu+2)(3\nu+5)a_2^3$$

$(\nu>4)$. Then, under a transformation which has a simple isolated fundamental point at the origin, (i) I is a relative invariant if $s=2$, $\nu>3$; (ii) I and J are both relative invariants if $s=2$, $\nu>4$; (iii) $a_1, \cdots, a_{\nu-1}$ are invariant if $s>2$.

J. A. Todd (Cambridge, England).

Godeaux, Lucien. Construction de surfaces algébriques de diviseur quelconque. Bull. Soc. Roy. Sci. Liége **17**, 220–224 (1948).

Sia ν un intero >1, $\rho=4\nu^2-3\nu+2$, $p=2\nu+1$, p numero primo. Nello $S_{\rho+p-1}$, nel quale $x_0, x_1, \cdots, x_\rho, y_1, y_2, \cdots, y_{p-1}$ sono coordinate proiettive omogenee, l'omografia H:

$$\frac{x_0'}{x_0}=\frac{x_1'}{x_1}=\cdots=\frac{x_\rho'}{x_\rho}=\frac{y_1'}{\epsilon y_1}=\frac{y_2'}{\epsilon^2 y_2}=\cdots=\frac{y'_{p-1}}{\epsilon^{p-1}y_{p-1}}$$

(dove ϵ è una radice p-esima primitiva dell'unità) è periodica

di periodo p. L'autore costruisce una superficie F d'ordine $2^{-r}\cdot(p!)^{p-1}$ dello S_{p+p-1}, che è trasformata in sè dalla H. Una superficie Φ, immagine della involuzione ciclica I^p, generata dalla H sulla F, si può ottenere proiettando la F dallo $S_{p-2}(y_1, y_2, \cdots, y_{p-1})$ sullo $S_p(x_0, x_1, \cdots, x_p)$. Per tale superficie Φ, che è d'ordine $p^{-1}2^{-r}(p!)^{p-1}$, si ha $\sigma=p$, essendo σ il divisore di Severi. Lo studio della Φ viene approfondito nel caso $p=5$. *F. Conforto* (Rome).

Godeaux, Lucien. Sur la détermination du système canonique de certaines surfaces algébriques. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **35**, 966–970 (1949).

Godeaux, Lucien. Remarques sur la construction de surfaces algébriques non rationnelles de genres zéro. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **35**, 971–975 (1949).

Godeaux, Lucien. Construction de surfaces algébriques irrégulières. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **36**, 14–22 (1950).

Amato, Vincenzo. Le curve algebriche a gruppo G_6. Matematiche, Catania **4**, 37–39 (1949).

Hohenberg, Fritz. Reelle birationale Strahlverwandtschaften im Raum als Bilder komplexer ebener Cremonatransformationen. Monatsh. Math. **53**, 324–335 (1949).

Verf. betrachtet im Strahlraum S zwei parallele Spurebenen E_1 und E_2, bestimmt die Schnittpunkte der reellen Geraden g mit E_1 und E_2, fasst E_1 und E_2 als Gausssche Zahlenebenen der komplexen Veränderlichen z_1 und z_2 auf und ordnet g den Punkt (z_1, z_2) einer komplexen Hilfsebene E zu. Kurven in E werden also auf Strahlenkongruenzen in S abgebildet. Mit Cremonatransformationen in E korrespondieren gewisse Strahltransformationen. Verf. betrachten i.B. die Strahltransformationen, welche den Affinitäten und den Ähnlichkeiten in E entsprechen. *O. Bottema.*

Differential Geometry

Goldoni, Gino. Sulle varietà applicabili. Atti Sem. Mat. Fis. Univ. Modena **3**, 205–206 (1949).

A simple derivation is given for necessary and sufficient conditions that two varieties in Euclidean hyperspace be applicable. *V. G. Grove* (East Lansing, Mich.).

Goldoni, Gino. Generalizzazione di un teorema di Mannheim-Beltrami. Atti Sem. Mat. Fis. Univ. Modena **3**, 207–209 (1949).

Let Q be a hyperquadric in Euclidean space S_n, and P a point on Q. Consider at P a system of mutually orthogonal lines l_r, $r=1, 2, \cdots, n$. The n intersections of l_r with Q, distinct from P, determine a hyperplane π. The paper considers the locus of the intersection C of π with the normal to Q at P. The locus of C is found to be a quadric or to be the hyperplane at infinity. *V. G. Grove.*

Vincensini, Paul. Sur un mode de représentation des surfaces. C. R. Acad. Sci. Paris **229**, 1114–1115 (1949).

This note describes some properties of a point correspondence between the points P and P' of two surfaces S, S', the point P' being the intersection with S' of the normal to S at P. *V. G. Grove* (East Lansing, Mich.).

Buzano, Piero. Cilindri di rotazione e curve sghembe. Univ. e Politecnico Torino. Rend. Sem. Mat. **8**, 145–158 (1949).

The conditions that a sphere have contact of various orders with a curve at one of its points, or that the curve lie on a sphere, are well known. In this paper a similar problem is considered in which the sphere is replaced by a cylinder of revolution. *V. G. Grove.*

Poivert, Jules. Les transformées en chaînes. Rev. Trimest. Canad. **35**, 184–195 (1949).

A plane conchoidal transformation τ is one in which every point, other than a fixed origin O, is displaced through a fixed distance and in the direction making a fixed angle with the radius vector to O. The author discusses the conditions under which a chain of curves $C, \tau C, \tau^2 C, \cdots$ tends to a limit curve K and the class of curves C for which the limit curve K is the same. The case discussed in detail is that of the limit curve, called the pelote, which arises when C is taken to be a line through O. *J. G. Semple.*

Löbell, Frank. Betrachtungen über Flächenabbildungen. VII. Bestimmung der einer gegebenen Fläche mit gegebenen Spreizvektor entsprechenden Flächen. S.-B. Math.-Nat. Kl. Bayer. Akad. Wiss. **1948**, 71–79 (1949).

[For part VI cf. the same S.-B. **1947**, 179–186 (1949); these Rev. **11**, 131.] The spread vector [Spreizvector] of the mapping of a surface with vector equation $x=x(u, v)$ onto the surface $y=y(u, v)$ is defined by the equation $J=(x_u\times y_v-x_v\times y_u)/(cx_ux_v)$, where c is the positive unit normal to surface x. The author considers the question of determining those surfaces y corresponding to a given surface x and for which the spread vector is a given function of position on x. The problem is reduced to the solution of a second order scalar partial differential equation. For this purpose the properties of the differential operator $D=[1/(cx_ux_v)][x_v\partial/\partial u-x_u\partial/\partial v]$ are developed, together with its relationship to a known tensor, or symmetric affinor Γ. *S. B. Jackson* (College Park, Md.).

Löbell, Frank. Betrachtungen über Flächenabbildungen. VIII. Bestimmung der einer gegebenen Fläche gleichmässig entsprechenden Flächen. S.-B. Math.-Nat. Kl. Bayer. Akad. Wiss. **1948**, 227–234 (1949).

[Cf. the preceding review.] Let the surface $x=x(u, v)$ in Euclidean 3-space be mapped on the surface $y=y(u, v)$. The projection measure [Rissmassstab] n and the transverse measure (Querrissmassstab) q are defined by the equations $n=(dx\cdot dy)/dx^2$, $q=(cdxdy)/dx^2$, where c is the positive unit normal to x. The quantities n and q are normally functions of both direction and position, but if they are independent of direction the mapping is called uniform. The problem here solved is the finding of those surfaces $y(u, v)$ for which the mapping from a given surface $x(u, v)$ is uniform. It appears that either n or q can be arbitrarily specified as a function of position on x, the solution then depending on the solution of a scalar second order partial differential equation. *S. B. Jackson.*

Löbell, Frank. Betrachtungen über Flächenabbildungen. IX. Flächen, die sich gegenseitig gleichmässig entsprechen. S.-B. Math.-Nat. Kl. Bayer. Akad. Wiss. **1948**, 335–339 (1949).

To a mapping of a surface $x=x(u, v)$ in Euclidean 3-space on a second surface $y=y(u, v)$ there corresponds an inverse mapping of y on x. The problem considered in this paper

is the determination of those pairs of surfaces for which both these mappings are uniform [cf. the preceding review for definition of uniform mappings]. *S. B. Jackson.*

Strubecker, Karl. Über die Flächen, deren Asymptotenlinien beider Scharen linearen Komplexen angehören. Math. Z. 52, 401–435 (1949).

A curve whose tangents all belong to one linear complex shall be called a screw line. The author deals with the surfaces Φ whose asymptotic curves of both families are screw lines. In the first part of the paper it is proved that the linear complexes associated with two asymptotic curves which do not belong to the same family are in involution. This fact is used to divide the surfaces Φ into three classes as follows. Let S_1 be the set of axes of those singular complexes which depend linearly on the set of linear complexes associated with one family of asymptotic curves on Φ. Let S_2 be defined similarly with respect to the other family. Then Φ, except for some trivial cases, satisfies one of the following conditions: (I) S_1 and S_2 together constitute a nondegenerate quadric, (II) S_1 and S_2 are linear pencils in distinct planes and with distinct vertices but with one common ray, (III) S_1 and S_2 are coinciding linear pencils. For these three classes the following metric characterizations are obtained: $\Phi_{\mathrm{I}}, \Phi_{\mathrm{II}}, \Phi_{\mathrm{III}}$ can be projectively embedded in an elliptic space of curvature 1, in a quasi-elliptic space and in an isotropic space, respectively, and in each of these spaces Φ is characterized by the fact that its relative curvature $1/R_1R_2$ is constant and equal to -1. The surfaces $\Phi_{\mathrm{I}}, \Phi_{\mathrm{II}}, \Phi_{\mathrm{III}}$ can be generated by means of a right- (left-) parallel displacement in the sense of Clifford of a curve of constant torsion 1 (-1) along a curve of constant torsion -1 (1) provided the two curves have a point in common and possess the same osculating plane at that point. Analytical representations free of quadratures are obtained for $\Phi_{\mathrm{I}}, \Phi_{\mathrm{II}}, \Phi_{\mathrm{III}}$. The two families of asymptotic curves coincide with the families of curves obtained by the displacements of the two original curves of torsion 1 and -1. Attention is paid to the classification of the surfaces Φ under projective transformations with real coefficients. The paper contains a history and a bibliography on the surfaces Φ.

W. van der Kulk (Providence, R. I.).

Mishra, R. S. Five families of ruled surfaces thro' a line of a rectilinear congruence. J. Indian Math. Soc. (N.S.) 13, 81–84 (1949).

Une congruence rectiligne étant définie par sa surface de départ et les deux formes quadratiques de Kummer $f=G_{\alpha\beta}du^\alpha du^\beta$, $\Phi=\mu_{\alpha\beta}du^\alpha du^\beta$, et J désignant le Jacobien de deux formes différentielles quadratiques, l'auteur effectue la recherche des surfaces réglées de la congruence dont les traces sur la surface de départ s'obtiennent en annulant le résultat d'une application répétée de l'opération J aux deux formes f et Φ. On obtient ainsi cinq familles de surfaces réglées de la congruence et cinq seulement, à savoir: les surfaces à représentation sphérique isotrope; les surfaces dont les lignes de striction sont sur la surface de départ; les surfaces principales; les surfaces de distribution; enfin une cinquième famille de surfaces caractérisées par le fait que, sur chaque rayon, la distance du point central au point de départ, est égale au rapport du produit t_1t_2 des distances de ce dernier point aux points limites de la congruence et de la distance $\frac{1}{2}(t_1+t_2)$ de ce même point au point moyen du rayon envisagé. Les équations de ces cinq familles de surfaces sont respectivement: $f=0$, $\Phi=0$, $J(f,\Phi)=0$, $J[f, J(f,\Phi)]=0$, $J[\Phi, J(f,\Phi)]=0$. *P. Vincensini.*

Chariar, V. R., and Singh, B. On a certain rectilinear congruence. J. Indian Math. Soc. (N.S.) 13, 148–151 (1949).

Si l'on considère, dans une congruence rectiligne, un pinceau de rayons limité par une surface réglée fermée et une trajectoire orthogonale des génératrices de cette surface, la distance p entre deux points consécutifs d'intersection de la trajectoire avec un même rayon est la même pour toutes les trajectoires orthogonales, et p est ce que R. Behari appelle le "pitch" du pinceau pour le rayon envisagé. Pour les congruences dont les rayons s'appuient sur deux courbes fixes $\Gamma(u)$ et $\Gamma'(v)$ on a pour p l'expression connue: $p=-\iint\partial^2x/\partial u\partial v$, où x est la distance des deux points M, M' où un rayon quelconque du pinceau coupe Γ et Γ'. Les auteurs donnent pour p la nouvelle expression $p=-\iint x^{-1}\{\cos\omega-\cos\theta\cos\phi\}dudv$, ou ω est l'angle des tangentes en M et M' à Γ et Γ', et θ, ϕ les angles de ces tangentes avec MM'. A titre d'application ils recherchent les congruences normales à nappes focales curvilignes, mais une légère omission dans l'interprétation du résultat les amène à dire que les seules solutions sont fournies par les congruences des droites s'appuyant sur un cercle et son axe, oubliant ainsi celles formées des droites s'appuyant sur deux coniques focales l'une de l'autre. *P. Vincensini.*

Blaschke, Wilhelm. Kinematische Begründung von S. Lie's Geraden-Kugel-Abbildung. S.-B. Math.-Nat. Kl. Bayer. Akad. Wiss. 1948, 291–297 (1949).

Let $p'\to P$ be a correspondence of points p' and straight lines P such that (1) to any straight line Q' (looked upon as a locus of p') corresponds a regulus $R(Q')$, and (2) if Q_1' and Q_2' intersect in a point, the corresponding reguli have a line in common. Then the correspondence $p'\to P$ is the Lie "sphere"-line contact transformation. The author establishes such a correspondence by combining two mappings: a mapping of line elements ϵ of a plane on lines P of the space $\epsilon\to P$ (this mapping being closely connected with the cyclographic mapping) and the so-called cinematic mapping of line elements ϵ on points p' of the space. The composition $p'\to\epsilon\to P$ is the desired Lie contact transformation. *V. Hlavatý* (Bloomington, Ind.).

Eisenhart, Luther P. Homogeneous contact transformations. Proc. Nat. Acad. Sci. U. S. A. 36, 25–30 (1950).

The following two theorems are proved. (1) If φ is any nonlinear function of $p_1, \cdots, p_n$, homogeneous of degree one, and ψ^i are any n independent functions of $x^1-\partial\varphi/\partial p_1, \cdots, x^n-\partial\varphi/\partial p_n$, the equations $\bar{x}^i=\psi^i$ are the first set of equations of a homogeneous contact transformation, and the second set is obtained by solving the first set of equations

$$\bar{p}_i\frac{\partial\psi^i}{\partial y^\alpha}=p_\alpha;\quad \bar{p}_i\frac{\partial\psi^i}{\partial y^\beta}\frac{\partial^2\varphi}{\partial p_\alpha\partial p_\beta}=0,$$

where $y^\alpha=x^\alpha-\partial\varphi/\partial p_\alpha$. (2) Given any nonlinear function φ of $p_1, \cdots, p_r$, homogeneous of degree one in the p's, and functions $\psi_i^\alpha(x^{r+1}, \cdots, x^n)$, for $\alpha=1, \cdots, r$; $i=1, \cdots, n$, such that the determinant D of ψ_i^α and $p_\alpha\partial\psi_i^\alpha/\partial x^\lambda$, for $\lambda=x^{r+1}, \cdots, x^n$, is not zero; a homogeneous contact transformation is defined by the equations $\bar{x}^i=f^i(\theta^1, \cdots, \theta^n)$ and $\bar{p}_i\partial f^i/\partial\theta^j=p_\beta\psi_j^\beta$, where f^i are any n independent functions of the θ's, the latter defined by

$$\theta^i(x,p)=(x^\alpha-\partial\varphi/\partial p_\alpha)\bar{\psi}_\alpha{}^i-p_\lambda\bar{\psi}^{\lambda i},$$

where $\bar{\psi}_\alpha{}^i$ and $\bar{\psi}^{\lambda i}$ are the cofactors of ψ_i^α and $p_\alpha(\partial\psi_i^\alpha/\partial x^\lambda)$, respectively, in D, divided by D. *J. A. Schouten.*

Sakellariou, Neilos. Invariants of simultaneous contact transformations. Prakt. Akad. Athēnōn 23 (1948), 47–51 (1949). (Greek)

The author examines in the three-dimensional space a multiplicity of elements of contact $[X]$ depending on a parameter ω. He considers two systems of orthogonal coordinates $[Ox_i, Ox_i^*]$ and two elements of contact X, X^*, of which the X^* is supposed to be on the perpendicular XT on the plane of the element X and at a distance $XX^*=C$. He is trying to find invariants of the multiplicity $[X]$ in the transformation $[(X), (X^*)]$. *P. Zervos* (Athens).

Su, Buchin. Contributions to the projective theory of curves in space of five dimensions. Univ. Nac. Tucumán. Revista A. 7, 15–79 (1949).

The notion of the representable singularity of a plane curve is the basic idea upon which this study depends. This notion was introduced by the author and used extensively by him in his earlier papers [Tôhoku Math. J. 45, 239–244 (1938); J. Chinese Math. Soc. 2, 139–151 (1940); Ann. Mat. Pura Appl. (4) 26, 177–197 (1947); these Rev. 2, 299; 10, 144]. Let O be a singular point of a plane curve C, such that the tangent t_O of C at O has contact of order $m-1$ ($\geqq 2$) with C. An algebraic curve C_m of order m can be determined having (1) an arbitrarily assigned point M, outside of t_O, for an $(m-1)$ple point with coincident tangents t and (2) contact of order $m+1$ with C at O. The tangent t meets t_O at a point O_{m+1}, independent of the point M. A singular point is called the representable singular point of order m when the point M can be so chosen that the curve C_m has contact of order higher than $m+1$ with C at O. This notion provides a means of constructing a system of "canonical pyramids" of reference at a generic point P of a curve Γ in S_5. Let γ_2 denote the curve of intersection of the developable hypersurface of Γ with the osculating plane S_2 of Γ at P. Let Γ_2 denote the osculating conic of γ_2 at P. To an assigned point P_1 on the tangent S_1 of Γ at P there corresponds a point P_2, the point of contact of the second tangent to Γ_2 which passes through P_1. There exists an $(m-2)$-parameter family of planes passing through S_1 and contained in the osculating space S_m of m dimensions of Γ at P, but not belonging to the osculating space S_{m-1} of $m-1$ dimensions at the same point, where $3\leqq m\leqq 5$. Each plane of this family intersects the developable hypersurface of Γ in a curve γ_m which has contact of order $m-1$ with S_1. Among this family of planes a one-parameter family can, therefore, be determined such that the section γ_m corresponding to each plane of the family has P for its representable singularity of order m. A unique plane of the one-parameter family can be determined by the condition that the corresponding point O_{m+1} be coincident with the assigned point P_1 on S_1. Having the plane constructed through S_1 for $m=3, 4, 5$ the corresponding point O_{2m}, which is denoted by P_m ($m=3, 4, 5$), is then obtained. A canonical pyramid $\{PP_1P_2P_3P_4P_5\}$ is thus determined if P_1 is preassigned on S_1. With respect to each canonical pyramid of reference the developments for the equations of Γ assume the forms of "canonical expansions." The effects upon the coefficients of these expansions of the collineations which transform a canonical pyramid of reference to another canonical pyramid are determined. Four linear differential forms, called elements of projective arc length of the first, second, third, and fourth species, are found which are invariant under these collineations. The author obtains, corresponding to each of these forms, (1) projective Frenet formulas expressed in normalized coordinates, and (2) a geometric determination of a "normal pyramid" of reference (in each case a special canonical pyramid). Let Q_j denote the points of intersection of the tangent at P to Γ with the hyperplane $x^j=0$, $j=0, 1, 2, 3, 4, 5$ ($Q_0=P$). Suppose that these points are arranged in the order $Q_0, Q_1, Q_2, Q_3, Q_4, Q_5$. There are fifteen cross-ratios of four points from these six which preserve the order of the points. The four fundamental differential forms are expressible as the principal parts of certain linear combinations of these fifteen cross-ratios.

P. O. Bell (Lawrence, Kan.).

Zammataro, Nicola. Su una proprietà metrica delle omografie fra spazi pluridimensionali. Matematiche, Catania 4, 64–66 (1949).

Tuganov, N. G. Affine-basic lines on a surface. Doklady Akad. Nauk SSSR (N.S.) 69, 499–502 (1949). (Russian)

Continuation of the study begun in two previous papers [same Doklady (N.S.) 57, 327–330 (1947); 58, 1911–1914 (1947); these Rev. 9, 201, 378]. Here it is shown that the mapping of one surface upon another preserving affine basic lines depends on 12 arbitrary functions of one parameter. The determination of surfaces one family of whose asymptotic lines is a family of basic lines of the second order depends on 6 arbitrary functions of one parameter.

M. S. Knebelman (Pullman, Wash.).

Nakae, Tatsuo. Sur un groupe de transformations d'éléments linéaires qui laissent $ds^2=g_{ij}dx^idx^j$ invariant. Mem. Coll. Sci. Kyoto Imp. Univ. Ser. A. 22, 455–458 (1939).

To every group of point transformations in X_n belongs a group of transformations of points and linear elements, the so-called extended group. If each transformation of this extended group is followed by a linear homogeneous transformation of linear elements at every point, leaving the points invariant, we get another group. If in V_n the extended group leaves $ds^2=g_{ij}dx^idx^j$ invariant and if the linear homogeneous transformations are orthogonal at all points the group constructed in this way also leaves ds^2 invariant. This seems clear to the reviewer without any calculation. The author gives the calculations. *J. A. Schouten* (Epe).

Walker, A. G. On parallel fields of partially null vector spaces. Quart. J. Math., Oxford Ser. 20, 135–145 (1949).

Un champ de r-plans tangents à une variété riemannienne V_n est dit parallèle lorsque, deux points P et Q quelconques de V_n étant choisis, tout vecteur du r-plan associé à P est transformé, par parallélisme le long d'un chemin quelconque reliant P à Q, en un vecteur du r-plan associé à Q. Les champs de 0-plans et de n-plans tangents fournissent des exemples triviaux de champs parallèles. Si V_n est douée d'une métrique indéfinie, un r-plan admet une "partie nulle" qui est l'intersection du r-plan et du $(n-r)$-plan conjugué. Le present papier, qui n'envisage qu'un point de vue local, est particulièrement destiné à mettre en évidence les différences entre le cas des champs parallèles sans partie nulle et celui d'un champ parallèle de r-plans dont la partie nulle ne se réduit pas à zéro. En particulier le théorème classique de locale réductibilité n'est plus vrai dans le second cas. La liaison des champs étudiés avec les tenseurs symétriques du second ordre, récurrents au sens de Ruse, fait l'objet de plusieurs théorèmes. À tout tenseur de cette espèce, non proportionnel à g_{ij}, correspond un champ parallèle non trivial. Mais inversement il peut exister des champs parallèles de plans partiellement nuls qui ne sont engendrés

par aucun tenseur de cette espèce. Il n'en est évidemment pas de même dans le cas de champs parallèles de plans sans partie nulle. *A. Lichnerowicz* (Paris).

Ruse, H. S. On parallel fields of planes in a Riemannian space. Quart. J. Math., Oxford Ser. 20, 218–234 (1949).

Les champs parallèles de plans partiellement nuls ont été discutés d'une manière générale par A. G. Walker [voir l'analyse ci-dessus]. L'auteur étudie principalement ici les particularités présentées par les champs parallèles partiellement nuls d'une variété riemannienne V_4. Il fait une analyse exhaustive des différents cas: champs parallèles de 2-plans nuls, de 1-plans nuls et champs parallèles de 2-plans semi-nuls. Il est remarquable de noter que l'existence d'un champ parallèle de 1-plans nuls (ou de 2-plans semi-nuls) implique l'existence de champs de 2-plans nuls. Il en résulte que, dans le cas ou les plans parallèles sont réels, le métrique de V_4 est localement réductible à la forme

$$ds^2=2dxdz+2dydt+\alpha dz^2+2\beta dzdt+\gamma dt^2$$

où α, β, γ sont des fonctions des quatre coordonnées. L'auteur annonce que cette forme canonique est un cas particulier de formes canoniques, trouvées par A. G. Walker et non encore publiées, pour la métrique des V_n à champs parallèles partiellement nuls. Il annonce aussi que l'existence dans une V_n de dimension paire d'un champ parallèle nul de dimension $(\frac{1}{2}n-1)$ entraîne l'existence de champs parallèles autres que les champs conjugués.

A. Lichnerowicz (Paris).

Tonolo, Angelo. Sulle varietà Riemanniane normali a tre dimensioni. Pont. Acad. Sci. Acta 13, 29–53 (1949).

The principal directions of the Ricci tensor $\mathfrak{R}_{\mu\lambda}=\mathfrak{R}_{\kappa\mu\lambda}^{\cdots\kappa}$ are not always V_3-normal. Necessary and sufficient conditions, for the special case that $\mathfrak{R}_{\mu\lambda}$ has rank 3 and 3 different eigenvalues, are well known and have the form (1) $\underset{c}{i}^{\nu}\underset{b}{i}^{\mu}\underset{a}{i}^{\lambda}\nabla_{\nu}\mathfrak{R}_{\mu\lambda}=0$; $a, b, c=1, \cdots, \mathrm{n}$; a, b, c unequal, where the $\underset{a}{i}^{\kappa}$ are unit eigenvectors. Specialists have long sought for invariant conditions not containing the eigenvectors. The author now gives these conditions. He introduces a tensor $\Phi^{\kappa\lambda}=\frac{1}{2}\mathfrak{J}^{\kappa\rho\sigma}\mathfrak{J}^{\lambda\tau\nu}\mathfrak{R}_{\rho\tau}\mathfrak{R}_{\sigma\nu}$, but this was not necessary because $\Phi^{\kappa\lambda}$ is to within a scalar factor equal to $\overset{-1}{R}{}^{\kappa\lambda}$. Using $\overset{-1}{R}{}^{\kappa\lambda}$, we may write his conditions in the form

$$R^{\tau}{}_{[\nu}\nabla_{\mu}R_{\lambda]\tau}=0, \tag{2}$$

$$\overset{-1}{R}{}^{\tau}{}_{[\nu}\nabla_{\mu}R_{\lambda]\tau}=0, \tag{3}$$

$$\overset{-1}{R}{}^{\tau}{}_{[\nu}\nabla_{\mu}\overset{-1}{R}_{\lambda]\tau}=0 \tag{4}$$

(of course there is a fourth equation but it is equivalent to the second). By transvection with $I^{\nu\mu\lambda}$ we get the three invariants that must vanish. The calculations are rather long, though the new conditions could have been derived from (1) in a few lines. It is highly probable that by using more modern methods a short proof, not using (1), can be given. The author does not seem to have observed that his conditions hold for any tensor of rank 3 with 3 different eigenvalues whether it is the Ricci tensor or not.

J. A. Schouten (Epe).

Tonolo, Angelo. Sulle varietà riemanniane normali a tre dimensioni. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 6, 438–444 (1949).

This is a short survey of the paper reviewed above.

J. A. Schouten (Epe).

Sorace, Orazio. Su alcune operazioni in una varietà di Riemann. Matematiche, Catania 4, 53–60 (1949).

Sorace, Orazio. Trasporti rigidi di vettori su di una varietà metrica di Riemann. Matematiche, Catania 4, 67–76, 106 (1949).

(I) Let $\underset{1}{\mathbf{i}}, \cdots, \underset{n-3}{\mathbf{i}}$ be $n-3$ vector fields consisting of mutually perpendicular unit vectors, parallel along a curve $L(t)$. Denote by $i^{\nu\mu\lambda}$ the complementary trivector (to the $(n-3)$-vector $\underset{[1}{\mathbf{i}}\cdots\underset{n-3]}{\mathbf{i}}$) constructed in the usual way by means of the Ricci tensor ϵ. Let S_λ be an arbitrary vector field along $L(t)$, $\dot{S}_\lambda$ its covariant derivative along $L(t)$ and $S_{\lambda\mu}=2S_{[\lambda}\dot{S}_{\mu]}$. Put $\underset{1}{S}{}^{\nu}=2i^{\nu\lambda\mu}S_{\lambda\mu}$ and $\underset{1}{S}_{\lambda\mu}=2\underset{1}{S}_{[\lambda}\underset{1}{\dot{S}}_{\mu]}$. Then it turns out that $\underset{2}{S}{}^{\nu}\equiv 2i^{\nu\lambda\mu}\underset{1}{S}_{\lambda\mu}$ may be expressed in the following way:

$$\underset{2}{S}{}^{\nu}=(-1)^{n-1}D\Big[S^{\nu}-\sum_{r=1}^{n-3}\underset{r}{i}{}^{\nu}(\underset{r}{\mathbf{i}}\cdot\mathbf{S})\Big],$$

where D is the determinant of $\underset{1}{\mathbf{i}}, \cdots, \underset{n-3}{\mathbf{i}}, \mathbf{S}, \dot{\mathbf{S}}, \ddot{\mathbf{S}}$ divided by $\sqrt{g}$, g being the discriminant of ds^2.

(II) A rigid motion of a unit vector u^{ν} along $L(t)$ is defined by $\dot{u}^{\nu}+A^{\nu}{}_{\lambda}u^{\nu}=0$, where $A_{\mu\lambda}$ is the "bivector of rotation." The author uses the results mentioned in (I) in order to find a necessary and sufficient condition (too complicated to be reproduced in a short review) that $A_{\mu\lambda}$ may be written $A_{\mu\lambda}=2v_{[\mu}\dot{v}_{\lambda]}$, v_μ being a unit vector.

V. Hlavatý (Bloomington, Ind.).

Vagner, V. V. Classification of simple differential-geometric objects. Doklady Akad. Nauk SSSR (N.S.) 69, 293–296 (1949). (Russian)

The problem of obtaining all possible types of differential geometric objects with N components in a space X_n is equivalent to the problem of obtaining all continuous transitive representations on the N-dimensional arithmetic space of the differentiable group $\mathfrak{D}^{(\nu,n)}$ defined as a transformation group of $\nu\cdot n$ variables $\xi^{(s)\alpha}$ $(s=1, \cdots, \nu)$:

$$\xi^{(s)\alpha}=\sum_{k=1}^{s}\frac{1}{k!}A^{\alpha}_{\alpha_1\cdots\alpha_k}\sum_{i_1+\cdots+i_k=s}\frac{s!}{i_1!\cdots i_k!}\xi^{(i_1)\alpha_1}\cdots\xi^{(i_k)\alpha_k}.$$

The group depends on $n\left[\binom{n+\nu}{n}-1\right]$ parameters $A^{\alpha}{}_{\alpha_1\cdots\alpha_k}$, symmetric in the lower indices. The equations $A^{\alpha}{}_{\alpha_1\cdots\alpha_k}=\delta^{\alpha}{}_{\alpha_1\cdots\alpha}{}^{k}$ $(k=1, \cdots, i<\nu)$ define a normal subgroup $\mathfrak{N}_i^{(\nu,n)}$ of $\mathfrak{D}^{(\nu,n)}$. Thus there is a sequence of normal subgroups $\mathfrak{N}_1\supset\cdots\supset\mathfrak{N}_{\nu-1}$, the last one being commutative. A geometrical object regarded as one in the tangent space defines a stationary subgroup $\mathfrak{H}$ of $\mathfrak{D}^{(\nu,n)}$. The object is of class u if $\mathfrak{H}$ contains $\mathfrak{N}_u$ but not $\mathfrak{N}_{u-1}$. If $\mathfrak{H}_1$ and $\mathfrak{H}_2$ are the stationary groups of two objects, the first is a function of the second if $\mathfrak{H}_1\supset\mathfrak{H}_2$. An object of class u is simple if there does not exist a function of it whose class is less than u but greater than 0. By investigating the structure of the Lie algebra corresponding to $\mathfrak{D}^{(\nu,n)}$ the author proves that simple geometrical objects of class greater than 2, if $n>1$, and of class greater than 3, if $n=1$, do not exist. For class 2 $(n>1)$ they are $\Gamma^{\alpha}{}_{\beta\gamma}$-symmetric affine connection, Γ_{β}-contracted connection and $\Pi^{\alpha}{}_{\beta\gamma}$-projective connection. For $n>1$ every simple geometric object of class greater than 1 is similar to either the affine connection γ or projective connection p.

M. S. Knebelman (Pullman, Wash.).

Vagner, V. V. Classification of simple differential-geometric objects. Uspehi Matem. Nauk (N.S.) 5, no. 1(35), 213–214 (1950). (Russian)

A brief résumé of the paper reviewed above.

M. S. Knebelman (Pullman, Wash.).

Haimovici, Adolf. Sur la géométrie d'un groupe généralisant celui des transformations parallèles. Disquisit. Math. Phys. 7, 121–137 (1948).

L'espace affine à trois dimensions étant rapporté à un repère déterminé près, un élément de contact est défini par les coordonnées x^i de son centre et les composantes ξ_i du vecteur covariant attaché à son plan. Sur ces éléments de contact, l'auteur introduit le groupe de transformations (*) $x'^i=b^i+a_j^i(x^j+\rho v^j)$, $\xi_i'=A_i^j\xi_j$, où la matrice (a_j^i) est unimodulaire, où (A_i^j) est la matrice inverse de (a_j^i) et où les v^i sont les composantes d'un vecteur $\mathbf{v}$. La transformation (*) est donc le produit d'une translation (définie par le vecteur $\rho\mathbf{v}$) par une transformation affine unimodulaire. Le but de l'auteur est l'étude des propriétés des multiplicités à 1 et 2 paramètres d'éléments de contact, invariantes par le groupe (*). Un système complet d'invariants est trouvé dans ces deux cas. Le cas particulier où la multiplicité à 2 paramètres est la multiplicité des éléments de contact tangents à une surface, est spécialement approfondi, ce qui conduit l'auteur à un résultat nouveau sur la géométrie des surfaces déterminées à une transformation par plans tangents parallèles près.

A. Lichnerowicz (Paris).

Varga, O. Affinzusammenhängende Mannigfaltigkeiten von Linienelementen, die ein Inhaltsmass besitzen. Nederl. Akad. Wetensch., Proc. 52, 868–874 = Indagationes Math. 11, 316–322 (1949).

Essayant de généraliser des propriétés connues des espaces de Finsler, l'auteur étudie à quelles conditions une variété d'éléments linéaires à connexion affine peut admettre un élément de volume intrinsèque qui soit ou bien relatif à un champ de vecteurs ou bien indépendant du champ de vecteurs. En chaque point ξ d'un certain domaine de la variété V_n, dont la variété V_{2n-1} étudiée est la variété des éléments linéaires tangents, donnons-nous un vecteur $v=r(\xi)$. S'il existe une densité q, de poids $+1$, dépendant de l'élément linéaire envisagé, des coefficients de la connexion affine et de leurs dérivées du premier ordre, la forme $qd\xi^1\cdots d\xi^n$ définit un élément de volume relatif au champ de vecteurs $r(\xi)$. Si la densité q peut être choisie indépendante de tout champ de vecteurs, l'élément de volume est dit absolu. L'auteur montre que si la variété à connexion affine considérée admet un parallélisme absolu, il existe certainement un élément de volume relatif au champ de vecteurs correspondant. Il met en évidence deux cas où il existe un élément de volume absolu. L'un est un cas particulier des variétés à parallélisme absolu précédentes. Dans l'autre cas, la variété n'est pas à parallelisme absolu mais constitue une généralisation exacte des espaces de Finsler à élément de volume absolu. Une étude des relations des variétés étudiées avec les variétés finsleriennes complète le papier.

A. Lichnerowicz (Paris).

Yano, Kentaro. Sur la théorie des espaces à hyperconnexion euclidienne. I. Proc. Japan Acad. 21 (1945), 156–163 (1949).

Étant donnée une variété numérique à n dimensions on peut attacher à chaque point de cette variété un espace tangent E_m à $m(>n)$ dimensions. Si la loi de raccord des espaces tangents attachés aux deux points infiniment voisins est définie par une transformation infinitésimale d'un groupe euclidien, l'espace s'appelle un espace à hyperconnection euclidien. Dans la présente note l'auteur étudie systématiquement ces espaces. Il part d'un espace V_n de Riemann dont le tenseur métrique est désigné par g_{ij} $(i,j=1,\cdots,n)$. Il suppose que l'espace E_m attaché à un point x^i contient l'espace tangent E_n qui est associé à ce point comme un point du V_n. L'espace E_m est doué d'un tenseur métrique fondamental $G_{\mu\nu}$ $(\mu,\nu=1,\cdots,m)$. Les changements du repère de E_m sont déterminés par une matrice $(A_\lambda^{\lambda'})$. Si la transformation de cordonnées de V_n est determinée par $x'^i=x'^i(x^1,\cdots,x^n)$, l'algèbre tensorielle peut se décrire par les quantités $A_\lambda^{\lambda'}$, $\partial x'^i/\partial x^i$, B_i^λ et B_P^λ $(P=n+1,\cdots,m)$. Les B_i^λ sont par rapport à l'indice λ, n vecteurs contrevariants situés dans E_n et par rapport à l'indice i, m vecteurs covariants de E_m. Les B_P^λ sont par rapport à l'indice λ, $m-n$ vecteurs contrevariants orthogonaux aux vecteurs B_i^λ. A l'aide de ces quantités l'auteur définit par $g_{PQ}=G_{\mu\nu}B_P^\mu B_Q^\nu$ le tenseur métrique pour l'espace E_{m-n} orthogonal à E_n. En utilisant les quantités introduites ci-dessus et quelques tenseurs qui en sont déduits algébriquement, puis les symboles de Christoffel appartenant au tenseur g_{ij} l'auteur peut définir l'hyperconnexion euclidienne en ajoutant les fonctions $\Gamma^\lambda_{\mu k}$ de x'^i. Ces fonctions se transforment pendant une transformation de coordonnées et un changement du repère dans E_m d'après $\Gamma^{\lambda'}_{\mu' k'}=A^{\lambda'}_\lambda(A^\mu_{\mu'}\Gamma^\lambda_{\mu k}+A^\lambda_{\mu',k})\partial x^k/\partial x^{k'}$. La dérivation covariante appartenante à cette connexion est définie d'une manière évidente. De plus l'auteur définit aussi une dérivation pour des quantités mixtes telles que B_i^λ et B_Q^λ. Dans les formules entrent outre les quantités $\Gamma^\lambda_{\mu k}$ et les symboles de Christoffel, les quantités Γ^P_{Qk}. Les Γ^P_{Qk} déterminent la connexion dans E_{m-n}.

O. Varga.

Yano, Kentaro. Sur la théorie des espaces à hyperconnexion euclidienne. II. Proc. Japan Acad. 21 (1945), 164–170 (1949).

En poursuivant le travail référé ci-dessus, l'auteur détermine le tenseur de courbure de V_n, de E_{m-n} et celui de E_m. On peut faire dériver d'une manière évidente ces tenseurs en appliquant deux fois le procédé de dérivation covariante. Les rélations établis par l'auteur entre ces tenseurs généralisent un résultat de Michal et Botsford. Ensuite l'auteur donne la définition des lignes des plus droites relativement à E_m. Étant donné un champ de vecteurs V^λ parallèles, le long d'une courbe de l'espace V_n, cette courbe possède la propriété désirée si la projection du champ donné sur le V_n consiste en vecteurs tangents à cette courbe. En partant de cette définition l'auteur fait dériver les équations differentielles pour une telle ligne. Une dernière notion est celle du V_n plan relativement à E_m. Quand il y a un déplacement par parallélisme de l'espace vectoriel E_n dans n'importe quelle direction, V_n est plan rélativement à E_m. Si le vecteur $B_i^\lambda dx^i/ds$ de E_m tangent à une géodesique de V_n se déplace toujours parallélement à lui-même le long de cette géodesique, V_n est géodesique rélativement à E_m. L'auteur établit pour chacun des deux cas les conditions analytiques. La théorie unitaire des champs d'Einstein et Mayer correspond au cas d'un V_4 géodesique rélativement à E_5. L'auteur établit dans le cas le plus général qu'il traite $(m=n-1)$ les équations qui correspondent aux équations des champs de cette théorie.

O. Varga (Debrecen).

Yano, Kentaro, and Takano, Kazuo. Conics in D. van Dantzig's projective space. Proc. Japan Acad. 21 (1945), 179–187 (1949).

In der Cartanschen Theorie der projektivzusammenhängenden Mannigfaltigkeiten sei das bewegliche Bezugssystem durch $[A_0,A_1,\cdots,A_n]$ und die infinitesimalen Transformationen desselben durch

$$dA_0=dx^0A_0+dx^iA_i;\quad dA_j=\Gamma^0_{jk}dx^kA_0+\Gamma^i_{jk}dx^kA_i$$

($i, j, k=1, 2, \cdots, n$) bestimmt. Die durch (1) $d^2(\rho A_a)/dt^2=0$ bestimmten Kurven bilden dann die natürliche Verallgemeinerung der Kegelschnitte in dieser Mannigfaltigkeit. Dabei ist ρ eine geeignete Funktion von t. Auf diesen Kegelschnitten kann ein projektiver Parameter eingeführt werden. Andererseits hat J. Haantjes mit Hilfe der van Dantzigschen krummlinigen homogenen Koordinaten x^λ ($\lambda=0, 1, \cdots, n$) für eine Mannigfaltigkeit einen projektiven Zusammenhang durch die Grössen $\pi^\lambda_{\mu\nu}$ erklärt. In dieser van Dantzig-Haantjesschen Theorie bezeichnen Verf. die durch (2) $\delta^2(\rho x^\lambda)/\delta t^2=0$ definierten Kurven als Kegelschnitte. Das Symbol δ bezeichnet eine mittels der $\pi^\lambda_{\mu\nu}$ erklärbare kovariante Ableitung und ρ eine geeignet gewählte Funktion des projektiven Parameters t. Verf. definieren nun die Komponenten Γ^0_{jk}, Γ^i_{jk} eines Cartanschen Zusammenhanges mit Hilfe der $\pi^\lambda_{\mu\nu}$. Bei Benützung dieses Zusammenhanges der Übertragungskomponenten kann als Endergebnis dieser Arbeit die Identität der durch (1) und (2) definierten Kegelschnitte, sowie der auf ihnen erklärten projektiven Parametern nachgewiesen werden.

O. Varga (Debrecen).

Iwamoto, Hideyuki. La géométrie des espaces métriques fondés sur la notion d'aire. I. Proc. Japan Acad. 21 (1945), 119–123 (1949).

L'auteur étudie la géométrie à connexion euclidienne défini sur une variété à N dimensions par l'intégrale multiple

$$O=\int L(x^i, p_\lambda^i)du^1\cdots du^k$$

($1\leqq k<N$) où $p_\lambda^i=\partial x^i/\partial u^\lambda$. Le tenseur métrique $g_{i_1\cdots i_k j_1\cdots j^k}$ est introduit, d'une manière analogue à Davies, comme polynome des variables L, $p_i^\lambda=L^{-1}\partial L/\partial p_\lambda^i$, $p_{ij}^{\lambda\mu}$, coïncidant dans le cas riemannien avec le tenseur métrique des k-vecteurs. Il coïncide naturellement dans le cas $k=N-1$ avec le tenseur introduit par É. Cartan. La dualité de Davies ne semble pas connue de l'auteur. Celui-ci se préoccupe particulièrement des conditions sans lesquelles le tenseur métrique précédent peut se déduire d'un tenseur g_{ij} par les équations $g_{i_1\cdots i_k j_1\cdots j_k}=g_{[i_1[j_1}\cdots g_{j_k]i_k]}$ classiques dans le cas riemannien.

A. Lichnerowicz (Paris).

Iwamoto, Hideyuki. La géométrie des espaces métriques fondés sur la notion d'aire. II. Proc. Japan Acad. 21 (1945), 223–226 (1949).

L'auteur généralise son étude précédent au cas où l'intégrale fondamentale est

$$O=\int L\left(x^i, \frac{\partial x^i}{\partial u^\lambda}, \cdots, \frac{\partial^M x^i}{\partial u^{\lambda_1}\cdots\partial u^{\lambda_k}}\right)du^1\cdots du^k.$$

Il retrouve en particulier les vecteurs de Synge de tous ordres et donne, dans un cas "général," des formules explicites définissant la connexion euclidienne, du sens de É. Cartan, intrinsèquement attachée à l'intégrale fondamentale.

A. Lichnerowicz (Paris).

Brickell, F. On the existence of metric differential geometries based on the notion of area. Proc. Cambridge Philos. Soc. 46, 67–72 (1950).

The n-dimensional differential geometry based on the idea of a two-dimensional area was introduced by Kawaguchi and Hokari [Proc. Imp. Acad. Tokyo 16, 313–319 (1940); these Rev. 2, 167] and later studied by several geometers. In this geometry the area of a two-dimensional plane element (x^i, u^{hk}) is defined by a fundamental function $L(x^i, u^{hk})$ of position x^i and of a simple bivector u^{hk}; L is supposed to be a positive homogeneous function of the first degree with respect to the variables u^{ij} and to possess continuous partial derivatives up to and including those of the fourth order. With these assumptions we can always define a bivector metric tensor $g_{hk,ij}$ which may be considered as

$$g_{hk,ij}=\partial^2(\tfrac{1}{2}L^2)/\partial u^{hk}\partial u^{ij}$$

in a certain sense [see a forthcoming paper of the reviewer; this fact will perhaps be proved also in a forthcoming paper by the author]. If the tensor $g_{hk,ij}$ splits up into the form $g_{hk,ij}=g_{hi}g_{kj}-g_{hj}g_{ki}$ with $g_{ij}=g_{ji}$, we say following R. Debever [Thèse, Bruxelles, 1947; these Rev. 9, 379] that the function L belongs to a class L_M. In the present paper it is shown that the existence of the class L_M is equivalent to the existence of functions g_{ij} satisfying the Pfaffian system $\frac{1}{2}u^{ij}dg_{hk,ij}=0$, where the differentials are with respect to u^{ij} only. The equations of this system are not all independent but at the most equivalent to the set of $2n-3$ equations. By use of the method of É. Cartan [Les systèmes différentiels extérieurs et leurs applications géométriques, Actual. Sci. Ind., no. 994, Hermann, Paris, 1945; these Rev. 7, 520], the present author shows that in the case $n=3$ the general analytic solution of the Pfaffian system depends on one arbitrary function but in the case $n=4$ no general analytic solution exists and the class L_M can arise only from singular solutions.

A. Kawaguchi (Sapporo).

NUMERICAL AND GRAPHICAL METHODS

*__Tables of the Bessel Functions of the First Kind of Orders Sixty-Four Through Seventy-Eight, by the Staff of the Computation Laboratory.__ The Annals of the Computation Laboratory of Harvard University, vol. XIII. Harvard University Press, Cambridge, Mass., 1949. x+566 pp. $8.00.

The present volume contains 10 decimal tables of $J_n(x)$ for $n=64(1)78$ and $x=0(.01)99.99$.

A. Erdélyi.

Williams, F. C., and Kilburn, T. A storage system for use with binary-digital computing machines. Proc. Inst. Elec. Engrs. Part III. 96, 81–98; discussion, 98–100 (1949).

Kilburn, T. The University of Manchester universal high-speed digital computing machine. Nature 164, 684–687 (1949).

The most important feature of the Manchester machine, its electrostatic storage, has been described elsewhere [see the preceding title]. Other aspects of the design are discussed here. The machine is a serial one working in the binary scale. Numbers have 40 digits, orders 20 digits. It has an electrostatic storage (3120 digits) and a magnetic drum storage (40560 digits) which are at present interconnected manually. Input and output are at present carried out manually and visually but teleprinter equipment will be installed. Certain logical operations, in addition to

those of addition, subtraction, multiplication, are built in but there is no direct facility for division; they are carried out at the rate of about 500 per second. The accumulator is a storage tube; another storage tube is used during the multiplication process and a third is used to provide a process of block modification of orders. The control of the machine is described in some detail. *J. Todd.*

Samuelson, P. A. Iterative computation of complex roots. J. Math. Physics 28, 259–267 (1950).

Ist ein Näherungswert a einer komplexen Wurzel eines Polynoms $f(x)$ gegeben, so ist nach Newton die Korrektion nach Newton gleich $-f(a)/f'(a)$. Verfasser berechnet nun $f(a)$ durch synthetische Division von $f(x)$ durch $h=(x-a)(x-\bar{a})$, eine Methode, die aber bekanntlich Collatz schon 1940 fand [Z. Angew. Math. Mech. **20**, 235–236 (1940); diese Rev. **2**, 61] und $f'(a)$ aus dem Quotienten durch nochmalige Division durch h, welche Methode zuerst von dem Rezensenten angewandt und leicht verallgemeinert wurde [Z. Angew. Math. Mech. **28**, 276–278 (1948); diese Rev. **10**, 152]. Es folgen einige Betrachtungen über die Newtonsche Methode und die Regula Falsi und eine Kritik der Methode von Lin. *E. Bodewig* (Den Haag).

Hammersley, J. M. The numerical reduction of non-singular matrix pencils. Philos. Mag. (7) **40**, 783–807 (1949).

The author extends the method of Frazer, Duncan and Collar [Elementary Matrices, Cambridge University Press, 1938] of computing the characteristic equation by replacing it by its minimum functions. The root-vectors are determined at the same time as the latent roots. By $x=Gz$ he transforms the characteristic equation $(T-\lambda E)x=0$ to the Frobenius rational canonical form $(T_1-\lambda E)z=0$, where $T=GT_1G^{-1}$ and the columns of G are the successive independent object and image points of a complete principal sequence. The difficulty of determining such a sequence is overcome by modifying G into H in which it is replaced by a set of arbitrary vectors. [There are some historical errors. E.g., what the author calls the method of Horst of 1935 is in reality the oldest method of all for determining the characteristic equation, namely that of Le Verrier [J. Math. Pures Appl. (1) **5**, 220–254 (1840), p. 230] which led Jacobi to develop a new method [J. Reine Angew. Math. **30**, 51–94 (1846)].] *E. Bodewig* (The Hague).

Plunkett, Robert. On the convergence of matrix iteration processes. Quart. Appl. Math. **7**, 419–421 (1950).

Specialising a method of Bückner for integral equations [Duke Math. J. **15**, 197–206 (1948); these Rev. **9**, 624] the author finds the well-known result that the usual iteration $v_k=u_0+Qv_{k-1}$ for the system $u=u_0+Qu$ of linear equations converges if all eigenvalues of Q lie within the circle of radius 1 around the origin. *E. Bodewig* (The Hague).

Mayot, Marcel. Sur la méthode d'intégration approchée de Tchebychef. C. R. Acad. Sci. Paris **230**, 429–430 (1950). Read the note on p.1138, Math. Rev. v.13.

The author shows that the method of Tchebychef is not practically applicable to the numerical integration of real functions if the number of ordinates to be used exceeds nine, since for a greater number the equation determining the abscissas may have complex roots. By calculation he found only two real roots in all cases from $n=10$ to 21 inclusive. *W. E. Milne* (Corvallis, Ore.).

Neugebauer, H. E. J. Ein einfaches Gerät zur Berechnung der Farbwertkoordinaten und anderer Stieltjesscher Integrale. Optik **6**, 8–13 (1950).

This describes an analog device for calculating integrals of the form $\int F(\lambda)g(\lambda)d\lambda$ which are of frequent occurrence in colorimetry. By proper choice of $\lambda_1, \lambda_2, \cdots$ (depending on g) this integral can be approximated in the form $\sum_\nu F(\lambda_\nu)$. The device is a refined measuring wheel for measuring the total length of the chosen ordinates on a graph of $F(\lambda)$. Naturally, the device would be useful for a relatively small number of fixed functions g, for which the λ_k have been determined, while $F(\lambda)$ could vary arbitrarily from problem to problem. *H. B. Curry* (State College, Pa.).

Huang, Su-Shu. A note on the variational method for the scattering problem. Physical Rev. (2) **76**, 1878–1879 (1949).

A modification of the variational principle for continuous spectra [L. Hulthén, Kungl. Fysiografiska Sallskapets i Lund Forhandlingar [Proc. Roy. Physiog. Soc. Lund] **14**, no. 21 (1944); these Rev. **6**, 111] is suggested. Taking

$$\mathfrak{L}=\int_0^\infty \psi(H-k^2)\psi dr \text{ with } H=-d^2/dr^2+V(r)$$

and varying from the exact solution of $(H-k^2)\psi=0$ we have (1) $\delta\mathfrak{L}=k\delta\eta$, η being the asymptotic phase of ψ. Choosing a trial function depending linearly on indeterminate parameters $c_1, \cdots, c_n$ and $\lambda=\tan\eta$ we fix an approximate solution by the equations $\partial\mathfrak{L}/\partial c_i=0$, $i=1, \cdots, n$; (2) $\partial\mathfrak{L}/\partial\lambda=k$. Equation (2) is intended to replace the equation $\mathfrak{L}=0$ in the original formulation. While this device gives the advantage of linear equations, it does not lead to a stationary phase value, since the condition $\mathfrak{L}=0$ has been given up. This explains the somewhat poor agreement of Huang's numerical phase shifts with the earlier values. The reviewer would also like to point out that equation (1) can be used for correcting the procedure so as to get stationary phases [cf. L. Hulthén, Ark. Mat. Astr. Fys. **35A**, no. 25 (1948), sect. 4; these Rev. **10**, 120]. This gives very little extra work and leads to an excellent agreement between the phase shifts, calculated by means of Huang's trial functions, and the earlier values. *L. Hulthén* (Stockholm).

Hamel, Georg. Zur Fehlerschätzung bei gewöhnlichen Differentialgleichungen erster Ordnung. Z. Angew. Math. Mech. **29**, 337–341 (1949). (German. English, French and Russian summaries)

In this paper there is derived a recursion formula for the coefficients of the dominating power series associated with the power series expansion of the solution of an ordinary differential equation of first order. Use of this series enables one to estimate the error committed by truncation of the power series solution. *W. E. Milne* (Corvallis, Ore.).

Conforto, Fabio. Neue Fortschritte in der numerischen Lösung der partiellen Differentialgleichungen der höheren Technik. Arch. Math. **2**, 135–138 (1950).

After some general comment on the progress of numerical methods for partial differential equations the author outlines briefly Picone's method as applied to the problems of elasticity. *W. E. Milne* (Corvallis, Ore.).

*van Wijngaarden, A. **Écoulement potentiel autour d'un corps de révolution.** Colloques Internationaux du Centre National de la Recherche Scientifique, no. 14, Méthodes de calcul dans des problèmes de mécanique, pp. 72–87. Centre National de la Recherche Scientifique, Paris, 1949.

This paper deals with a modification of von Kármán's method [Abh. Aerodynamischen Inst. Tech. Hochschule Aachen **6**, 1–17 (1927)] for calculating the flow of an inviscid incompressible fluid about a body of revolution. The method of von Kármán consists of formulating the problem as an integral equation which is reduced to a system of linear simultaneous equations by the assumption that the unknown function is a step-function. The author suggests that this reduction be made by assuming that the unknown function is a continuous piecewise linear function and gives tables to facilitate the application of his method.

C. Saltzer (Cleveland, Ohio).

Zernike, F. The differential analyzer as model of a continuous machine. Nederl. Tijdschr. Natuurkunde **15**, 265–274 (1949). (Dutch)

This is a lecture delivered before a symposium on modern calculating machines, held at Amsterdam in May, 1949. It gives a survey of the operations of a differential analyzer and of the technique of preparing problems for it. In the latter connection a simplification of the technique of Bush and Caldwell [J. Franklin Inst. **240**, 255–326 (1945); these Rev. **7**, 339] is given, in that it is not necessary to draw parallel lines for the various quantities. At the end there is a discussion of some sources of error in the integration. One of these, although not emphasized by Bush and his coworkers, was pointed out by Hele Shaw [Philos. Trans. Roy. Soc. London. Ser. A. **176**, 367–402 (1886)].

H. B. Curry (State College, Pa.).

Macnee, A. B. An electronic differential analyzer. Research Laboratory of Electronics, Massachusetts Institute of Technology, Tech. Rep. no. **90**, ii+43 pp. (1948).

This report describes a new differential analyzer designed to have moderate accuracy (1–10 per cent), low cost, high speed, extreme flexibility, and ability to handle all sorts of ordinary differential equations. It can thus be for quick, exploratory calculations. It is completely electronic. The output appears as a curve on the screen of a cathode-ray oscilloscope. Circuits for adding, integrating, multiplying and generating functions are described in the report; the multiplier and function generator use cathode ray tubes in combination with photoelectric cells. The report contains also a discussion of limitations on accuracy; and treatments of a number of special problems.

H. B. Curry.

Thomas, Joseph Miller. Nomographic disjunction. Duke Math. J. **16**, 419–432 (1949).

This paper gives a practical, self-contained and comprehensive discussion of the problem of representing a function $f(x, y, z)$ as a disjoint, i.e., as a determinant of Massau satisfying certain conditions which exclude cases devoid of nomographic interest. After introducing appropriate concepts mainly concerned with linear dependence [similar to those used by P. V. Nikolaev, Doklady Akad. Nauk SSSR (N.S.) **67**, 421–423 (1949); these Rev. **11**, 406] the author shows how the elements of a disjoint can be found if $f(x, y, z)$ is a nomographic function (i.e., equal to a disjoint). Conditions insuring that $f(x, y, z)$ is nomographic are not explicitly formulated since it is believed that the method explained can be applied to a given function and the equality of the function to the resulting determinant tested more easily than such conditions could be applied. The method, which differs according as the genus is zero or positive, avoids the use of derivatives; its relations to the work of others is explained.

R. Church (Annapolis, Md.).

ASTRONOMY

Popov, Kiril A. On the motion of the earth around its center of gravity. Doklady Akad. Nauk SSSR (N.S.) **69**, 755–758 (1949). (Russian)

The equations of Poisson: $d\theta/dt=-m\cos(\psi+l_\odot)$, $d(\psi+l_\odot)/dt=n+m\cot\theta\sin(\psi+l_\odot)$, giving the influence of the solar attraction on the earth's rotation, may be transformed by substituting $\mathfrak{x}=\sin\theta\sin(\psi+l_\odot)$, $\mathfrak{y}=-\sin\theta\cos(\psi+l_\odot)$, $\mathfrak{z}=\cos\theta$, $d\tau=ndt$. Then the new variables satisfy integral equations of Volterra's type and rapid convergence is ensured if the following units are used: the sun's mass, the sun-earth distance and the mean solar day.

W. Jardetzky (New York, N. Y.).

Mihailović, Dobrivoje. Sur les directions des mouvements instantanés relatifs dans le problème d'entre-choc de trois corps de la mecanique céleste. Bull. Soc. Math. Phys. Serbie **1**, 11–16 (1949). (Serbian. Russian and French summaries) M,2

Using his vectorial version of Sundman's and Block's treatment of the three-body problem, the author proves that Milankovitch's result in a special case, according to which the instantaneous directions of the relative motions of the three bodies intersect in one point, is valid also in the "collision" case.

L. Jacchia (Cambridge, Mass.).

Mihaïlovitch, D. Les intégrales premières dans le problème d'entrechoc de trois corps. Bull. Soc. Math. Phys. Serbie **1**, 45–61 (1949). (Serbian. Russian and French summaries) M,1

This paper is the expanded vectorial treatment of a problem which was investigated by Sundman and Block using scalar methods. It follows the lead of Milankovitch, who showed that the coincidence of the gravitational pole (the center of attraction) with the center of inertia implies the existence of exact solutions in the general three-body problem. Much of the analysis is made possible by the use of the independent variable $\tau=C(t_0-t)^{\frac{1}{3}}$ in place of the time t; C is a constant which is positive or negative according to whether the motion of the three bodies is centripetal or centrifugal. Starting from Sundman's result that the projections of the radii vectors are proportional to τ^2, the author shows that the vectors representing the forces are directed toward the center of inertia and thus, using Milankovitch's theorem, is able to conclude that the positions of the bodies correspond to one of Lagrange's configurations, a result found also by Sundman. Block's results in the cases of the equilateral and linear Lagrangian configurations are also reproduced using the same method.

L. Jacchia (Cambridge, Mass.).

Bilimović, Anton. Pfaff's expressions and the vector differential equations of planetary perturbations. Glas Srpske Akad. Nauka **191**, 83–115 (1948). (Serbian. Russian summary)

In previous papers the author showed that his so-called "Pfaff method," in which the canonical equations of motion are derived from Pfaff's expression, can be applied to various problems of mechanics. Here he applies his method to the problem of planetary perturbations and shows that the fundamental equations can be obtained directly in vectorial form. The resulting differential equations establish a relation between two independent arbitrary vector constants of the two-body problem and the partial gradients of the perturbing function along these vectors. The variation of the vectorial elements of motion can be obtained directly by integrating these differential equations. *L. Jacchia.*

Hil'mi, G. F. The virial theorem in celestial mechanics. Doklady Akad. Nauk SSSR (N.S.) **70**, 393–396 (1950). (Russian)

After stating that the treatments of the virial theorem to be found in the literature do not give a sufficiently general formulation for the usual applications made, the author gives such a formulation and its proof. However, the author's theorem was stated and proved by Jacobi [Vorlesungen über Dynamik, 2d ed., Reimer, Berlin, 1884, pp. 27–29] and may also be found, for example, in Wintner's book [The Analytical Foundations of Celestial Mechanics, Princeton University Press, 1941, p. 250; these Rev. **3**, 215].

R. G. Langebartel (Urbana, Ill.).

Baženov, G. M. Investigation of the convergence of iteration processes in the problem of the determination of orbits. Akad. Nauk SSSR. Bull. Inst. Teoret. Astr. **4**, no. 5(58), 207–225 (1949). (Russian)

The author demonstrates the advantages of employing a combination of Newton's method with the iteration method in the determination of an orbit by the schemes of Gauss and Harzer. Conditions for the convergence of the iteration process in the Gauss method are obtained.

R. G. Langebartel (Urbana, Ill.).

Proskurin, V. F. On the possibility of representing the motion of a satellite of Jupiter by the analytical theory of Brown. Akad. Nauk SSSR. Bull. Inst. Teoret. Astr. **4**, no. 4(57), 169–205 (1949). (Russian)

A comparison of Brown's theory of Jupiter's eighth satellite with observations shows residuals which, even after a redetermination of the arbitrary constants, reach $2^m.5$ in right ascension and $32'$ in declination. These discrepancies may be attributed either to inaccurate initial conditions or to incompleteness of the tables of motion. Corrected values for the initial conditions are computed by least squares and introduced into the tables of motion, but even after this extremely laborious procedure, the residuals cannot be reduced to less than $0^m.66$ in right ascension and $11'.2$ in declination for the period 1908–1938. The agreement is a little better from 1908 to 1935, when the residuals in right ascension reach a maximum of $0^m.4$. After 1935 they increase rapidly, reaching $1^m.05$ in 1942 and even greater values for more recent years. "Thus our attempt to improve the theory through a correction of the initial constants shows that it is impossible to bring the theory in satisfactory agreement with observations over a prolonged period of time. Evidently the incompleteness of the tables of motion is the basic cause of the imperfection of the theory." The author's final conclusion is that any future attempt should be in the direction of working out new methods capable of representing the complicated motion of Jupiter's VIII satellite, rather than trying to improve the old theory.

L. Jacchia (Cambridge, Mass.).

Stehle, P. The dynamics of star streaming. Astrophys. J. **110**, 250–260 (1949).

In this paper the problem in stellar dynamics [cf. S. Chandrasekhar, Principles of Stellar Dynamics, University of Chicago Press, 1942, chapters III and IV; these Rev. **4**, 57] of specifying the conditions under which a dynamical system will admit an integral (other than the energy integral) which is quadratic in the velocities is reconsidered by the methods of the tensor calculus. Denoting the quadratic form by $Q+\sigma=a_{ij}(u^i-u_0^i)(u^j-u_0^j)+\sigma$ (where the coefficients of the quadratic form a_{ij}, the motions of the local centroids u_0^i and the density function σ are functions of x^i and t) the author shows that, in the time independent case, the equations governing a_{ij} are (1) $a_{ij,k}+a_{ki,j}+a_{jk,i}=0$, (2) $\Delta_{i,j}+\Delta_{j,i}=0$ $(\Delta_i=a_{ij}u_0^j)$, (3) $\Delta_i V_{,}{}^i=0$, and (4) $\epsilon^{pij}(a_{il}V_{,}{}^l)_{,j}=0$. In the foregoing equations the comma preceding a subscript denotes covariant differentiation with respect to that subscript; also ϵ^{pij} denotes the alternating tensor. Equations (2) imply that the vector Δ_i defines a motion of the space into itself. In three-dimensional Euclidean space this means that Δ_i can be expressed as the sum of a translation and a rotation about the direction of translation. If the z-axis is taken as the direction of translation, $\Delta_1=\beta y$, $\Delta_2=-\beta x$, $\Delta_3=k$ (β and k constants). The corresponding solution of equation (3) is $V=V(x^2+y^2, z+(k/\beta)\tan^{-1}y/x)$, $V=V(x,y)$, $V=V(x^2+y^2,z)$ and $V=V(x,y,z)$, for the cases of both rotation and translation, translation only, rotation only, and no motion, respectively.

Expressing the vector Δ_i in terms of the generators $\eta_i^\alpha=\delta_i^\alpha$, $\xi_i^\alpha=-\epsilon(\alpha ij)x^j$ ($\alpha=1, 2, 3$ labels separate vectors and does not indicate tensor components for which Latin letters are reserved; this remark applies to all Greek indices) for translation and rotation of the Euclidean group of motions, the author next shows that the solution of equations (1) is represented by

$$a_{ij}=\alpha_{\sigma\rho}\eta_i^\sigma\eta_j^\rho+\beta_{\sigma\rho}(\eta_i^\sigma\xi_j^\rho+\eta_j^\sigma\xi_i^\rho)+\gamma_{\sigma\rho}\xi_i^\sigma\xi_j^\rho,$$

where the coefficients $\alpha_{\sigma\rho}$, $\beta_{\sigma\rho}$ and $\gamma_{\sigma\rho}$ are subject to the conditions $\alpha_{\sigma\rho}=\alpha_{\rho\sigma}$; $\beta_{\sigma\rho}=\beta_{\rho\sigma}$; $\sum_\sigma\beta_{\sigma\sigma}=0$. (There are twenty constants in all.)

In the last section of the paper the "compatibility condition" (4) is discussed. [While most of the results obtained in this paper have been known before, the presentation given in this paper exhibits the compactness and elegance of the tensor notation.] *S. Chandrasekhar.*

Huang, Su-Shu. A note on the mean square velocity in stellar statistics. Proc. Nat. Acad. Sci. U. S. A. **36**, 67–72 (1950).

This paper is devoted to the following problem. Given a distribution of velocities $f(v; l, m, n)$ such that the mean square velocity in a direction whose direction cosines are l, m and n with respect to an orthogonal set of axes (u_1, u_2, u_3) is given by

$$\overline{v^2}=\int_{-\infty}^{\infty} f(v; l, m, n)v^2dv=l^2\overline{u_1^2}+m^2\overline{u_2^2}+n^2\overline{u_3^2},$$

what are the mean square values of the quantities $v_1=v\cos\theta$, $v_2=v\sin\theta\cos\phi$ and $v_3=v\sin\theta\sin\phi$, where θ and ϕ are the

polar angles with respect to a set of axes (v_1, v_2, v_3) different from (u_1, u_2, u_3). In evaluating the means of $\overline{v_1^2}$, etc., the averaging is to be effected not only over v but also over all directions θ and ϕ. The author obtains the required moments in terms of $\overline{u_1^2}$, $\overline{u_2^2}$ and $\overline{u_3^2}$ and the relative orientation of the axes (v_1, v_2, v_3) and (u_1, u_2, u_3). Thus

$$\frac{1}{15}\overline{v_1^2}=(1+2\cos^2\vartheta_1)\overline{u_1^2}+(1+2\sin^2\vartheta_1\sin^2\varphi_1)\overline{u_2^2}+(1+2\sin^2\vartheta_1\cos^2\varphi_1)\overline{u_3^2},$$

where ϑ_1 and φ_1 are the polar angles of the v_1-axis in the coordinate system (u_1, u_2, u_3). *S. Chandrasekhar.*

Holopov, P. N. A numerical method for calculating the spatial density of stars in a spheroidal star cluster. Akad. Nauk SSSR. Astr. Zhurnal **26**, 298–304 (1949). (Russian)

In a previous paper [same vol., 110–114 (1949); these Rev. **10**, 746] the author outlined a procedure, based on a generalization of von Zeipel's method, for evaluating stellar densities in spheroidal globular clusters. The procedure proved to be very laborious and time consuming when it came to actual computation. The present paper is an attempt to solve the same problem using a different approach. Instead of von Zeipel's method, the starting point is here a method first outlined by Wallenquist in 1933 for the case of spherical clusters. By using star counts within coaxial similar ellipses instead of circular rings, the equation for the density f_m within two ellipsoids corresponding in space to two limiting ellipses reduces to the equation obtained by Wallenquist, except for a coefficient which involves the eccentricity and the inclination of the axis of revolution with respect to the line of sight. Numerical coefficients intended to facilitate the computation in the standard case of 25 concentric ellipses are given at the end of the paper. *L. Jacchia* (Cambridge, Mass.).

Prasad, Chandrika. Anharmonic pulsations of two particular models. Astrophys. J. **110**, 375–381 (1949).

The anharmonic pulsations of stellar models are considered for two different cases: the homogenous model and the model in which the density varies inversely as the square of the distance from the center. It is shown that overtones are absent in the homogenous model. The skewness of the velocity curve is therefore only due to self-coupling in this case and is very small. In the second case the skewness is still smaller and in the direction opposite to that observed. This is due to alternate signs of the effects of successive modes. The conclusion is that the velocity curve is seriously influenced by the model of the pulsating star. It should therefore be possible to use the observed velocity curve as an indication of the model. *G. Randers* (Oslo).

Sen, Hari K. The radiative equilibrium of a spherical planetary nebula. Astrophys. J. **110**, 276–287 (1949).

In this paper the equation of radiative transfer in a planetary nebula is reconsidered. The treatment differs from the earlier ones [cf. V. A. Ambarzumian, Pulkova Astr. Observ. Izvestiya **13**, no. 114, 1–26 (1933); S. Chandrasekhar, Z. Astrophys. **9**, 266–289 (1935)] in that allowance is made for the curvature of the layers. The case when the radial optical thickness (measured from $r=\infty$ inward) varies as some power of r is considered. However, the equation to be treated is similar to the standard one in which there are no sources; indeed the method of solution followed in this paper is exactly the same as in the standard case [cf. S. Chandrasekhar, same J. **101**, 95–107 (1944); these Rev. **6**, 244]. *S. Chandrasekhar* (Williams Bay, Wis.).

RELATIVITY

Scherrer, W. Über den Einfluss des metrischen Feldes auf ein skalares Materiefeld. Helvetica Phys. Acta **22**, 537–551 (1949).

This paper points up vividly a fundamental inconsistency in the usual treatment of the general theory of relativity. The issue is essentially the same as one on which A. N. Whitehead often insisted [e.g., The Principles of Relativity, Cambridge University Press, 1922, p. 83], namely, that by making the structure of space-time contingent upon matter, Einstein leaves the theory of measurement in a logically impossible situation. We are not able to measure where anything is until we know where everything is. To say that in the absence of matter, space-time is pseudo-Euclidean and so we can proceed by approximation is logically unsound since if matter determines the structure of space-time then in the absence of matter it should have no structure at all.

Scherrer takes this last remark as his starting-point and introduces a scalar matter-field described by ψ^2 which gives the distribution of particles per unit volume. From the variation problem

$$\delta\int(R\psi^2+4\omega g^{\rho\sigma}\psi_{,\rho}\psi_{,\sigma})dv=0,$$

where R is the curvature invariant, ω is a constant, dv is the four-dimensional space-time element, and the comma denotes ordinary differentiation, the author derives the field-equations

$$(R_{\rho\sigma}-\Lambda g_{\rho\sigma})\chi+\chi_{|\sigma\rho}+\tfrac{1}{2}g_{\rho\sigma}g^{\alpha\beta}\chi_{|\alpha\beta}+(\omega/\chi)\chi_{|\rho}\chi_{|\sigma}=0,$$

where Λ is a constant, the vertical bar indicates covariant differentiation, and $\chi=\psi^2$. This equation has the desired property of disappearing in the absence of matter. The author solves this equation for a spherically symmetric field obtaining a solution which reduces to the familiar Schwarzschild solution when ψ represents a single matter point. The energy tensor for the spherically symmetric case is calculated.

The paper concludes by indicating the desirability of solving the above equation for the two-body problem and by raising the question of why quantization was not necessary in the problem solved above. No reference is made to the series of more than forty papers in J. Sci. Hiroshima Univ. Ser. A beginning with that of Mimura [**5**, 99–106 (1935)] which are based on the same idea of introducing a matter-field. In the Japanese papers, ψ is a four-vector and the fundamental variation problem from which the field equations are derived differs from the above so that the Japanese papers do not treat exactly the same problem as that of Scherrer. *A. J. Coleman* (Toronto, Ont.).

Petrova, N. M. On the equations of motion and the mass tensor for systems of finite mass in the general theory of relativity. Akad. Nauk SSSR. Zhurnal Eksper. Teoret. Fiz. **19**, 989–999 (1949). (Russian)

This paper discusses the problem of determining the motion of n finite bodies by means of Einstein's gravitational equations. As in the fundamental paper of Einstein, Infeld and Hoffman [Ann. of Math. (2) **39**, 65–100 (1938)],

the method followed is one of successive approximation to the solution of the field equations. However, this paper is more closely related to one by V. A. Fock [Akad. Nauk SSSR. Zhurnal Eksper. Teoret. Fiz. **9**, 375–410 (1939)] which it refines in two details: (a) the correction due to general relativity theory of the Newtonian equations for n bodies is carried to a second approximation, (b) an approximation for the spatial components of the energy-momentum tensor for the same problem is obtained. The author, who makes no reference to the paper of Lichnerowicz on the same subject [J. Math. Pures Appl. (9) **23**, 37–63 (1944); these Rev. **7**, 266] claims that Fock's method has two advantages on that of Einstein, Infeld and Hoffman. First, the latter by treating matter as a point singularity are unable to determine the energy-momentum tensor. Secondly, Fock's use of "harmonic" coordinates in which $((-g)^{\frac{1}{2}}g^{\mu\nu})_{,\nu}=0$ greatly simplifies the calculations.

However, for Fock and Petrova, harmonic coordinates are not merely a labour-saving device but involve a matter of principle on which they take issue with Einstein and Infeld [Ann. of Math. (2) **41**, 455–464 (1940); these Rev. **1**, 283]. Petrova writes, "This system of coordinates (harmonic) . . . play in the general relativity theory the role of inertial coordinates in flat space. . . . Guided by the idea of equal rights for all coordinate systems in general relativity theory, they [Einstein, Infeld and Hoffman] consider the problem of finding invariant equations independent of the choice of coordinates. The assumption that all coordinate systems have equal rights does not appear to be correct. As Fock has shown, the harmonic coordinate system is preeminent over all others. The choice of any other coordinates leads to much more involved calculation but further, the motion of the particles in the new system cannot be interpreted as instantaneous motion of a particle in space. If our equations are evaluated in the case of a system of two particles they coincide with those of Einstein, Infeld and Hoffman, which means that, to the given approximation, the system of coordinates chosen by these authors is harmonic." More data on the problem of coordinates are contained in the recent paper of Einstein and Infeld [Canadian J. Math. **1**, 209–241 (1949); these Rev. **11**, 59] which Petrova had naturally not seen. Fortunately it is the duty of a reviewer merely to state and not to resolve such issues. However, it is perhaps worth suggesting that if Fock is right, the existence of a special "inertial" coordinate system, in which alone Newton's laws are valid in the first approximation, would provide a method of resolving the vexed problem of the apparent absoluteness of rotation as evidenced by Foucault's pendulum. *A. J. Coleman.*

Narlikar, V. V., and Prasad, Ayodhya. The Doppler effect in the field of a thick spherical shell. Proc. Indian Acad. Sci., Sect. A. **30**, 181–183 (1949).

The authors discuss the relativistic field of a thick spherical shell of inner and outer radii a, b. The boundary conditions are used to obtain the metric in the three parts of the field, the results differing from those of Combridge [Philos. Mag. (7) **1**, 276–279 (1926)]. In particular, for the region outside the shell ($r\geqq b$), the authors obtain the Schwarzschild line-element (for a central particle), and for the cavity ($0\leqq r\leqq a$) they obtain a Galilean ds^2. To the latter fact they call special attention, believing it to provide the only known example of a Galilean pocket in a more general space-time. Finally they deduce that, if an observer is situated within the shell and a light-source outside, the observer experiences a Doppler violet-shift arising from the continuity-conditions at the boundaries of the shell. *H. S. Ruse* (Leeds).

Curtis, A. R. The velocity of sound in general relativity, with a discussion of the problem of the fluid sphere with constant velocity of sound. Proc. Roy. Soc. London. Ser. A. **200**, 248–261 (1950).

The propagation of sound waves is investigated for cosmological models of general relativity and for different types of Schwarzschild interior metrics of spherically symmetric fluid spheres. The velocity of sound is found to be $(dp/dT_0^0)^{\frac{1}{2}}$ which has an upper bound of $3^{-\frac{1}{2}}$ times the velocity of light. Fluid spheres in which the velocity of sound is constant are investigated, particular attention being paid to the special case of an incompressible fluid. *A. Schild.*

Klein, O. On the thermodynamical equilibrium of fluids in gravitational fields. Rev. Modern Physics **21**, 531–533 (1949).

A fluid at rest in a static gravitational field is considered. A thermodynamic condition of equilibrium is obtained for each substance in the fluid. For the component of unordered temperature radiation the condition $(g_{44})^{\frac{1}{2}}T=$constant results; this was first discovered by Tolman [Relativity, Thermodynamics and Cosmology, Oxford, 1934, p. 318]. For a substance with nonvanishing chemical potential α, the additional relation $(g_{44})^{\frac{1}{2}}\alpha=$constant is obtained; this is a generalization of the ordinary barometer formula. Applications are indicated to the cases of (1) a cold Fermi gas of zero rest mass and (2) unordered temperature radiation, in spherically symmetric static gravitational fields. *A. Schild* (Pittsburgh, Pa.).

Trocheris, M. G. Electrodynamics in a rotating frame of reference. Philos. Mag. (7) **40**, 1143–1154 (1949).

The author deduces transformation functions which reduce to the Lorentz transformation in the limit of large distances from the axis of rotation. These functions are then applied to the theory of Sagnac's experiment on the time difference of two light beams traversing a closed circuit in opposite directions [C. R. Acad. Sci. Paris **157**, 708–710 (1913)]. Also modified Maxwell's equations, appropriate to a rotating system, are derived and their consequences discussed. *C. Kikuchi* (East Lansing, Mich.).

***McVittie, G. C. Cosmological Theory.** 2d ed. Methuen & Co., Ltd., London; John Wiley & Sons, Inc., New York, N. Y., 1949. viii+103 pp. $1.50.

Reprint, with correction of minor errors and misprints, of the 1937 edition. The first three of the five chapters present a rapid survey of our knowledge of extra-galactic nebulae, of the tensor calculus and of the principles of the general theory of relativity. These are applied in the fourth chapter to the problem of the expanding universe; McVittie's treatment of Hubble's data leads to a hyperbolic universe in which the rate of expansion is being retarded, in contrast with the small closed universe deduced by Hubble and Tolman. The booklet concludes with a brief account of E. A. Milne's kinematical theory of the universe. *H. P. Robertson* (Pasadena, Calif.).

Hönl, H. Zwei Bemerkungen zum kosmologischen Problem. Ann. Physik (6) **6**, 169–176 (1949).

(1) The author rediscovers that, in a homogeneous-isotropic expanding universe of "radius" $R(t)$, the momen-

tum p of a freely moving particle is inversely proportional to R. Implications of this for photons, cosmic rays and stellar aggregations are discussed. (2) He shows that the numerical value 1.9×10^{-17} sec^{-1} of Hubble's constant, when introduced into Einstein's original field equations (in which the cosmological constant $\lambda=0$), implies an age ($<1.7\times10^9$ yr) which is inconsistent with the ages (up to 4×10^9 yr) of known rocks and meteorites. He suggests that the most plausible way out of this discrepancy is to be found in modifying the laws of radioactive decay, as in the speculations of Jordan and/or Dirac. *H. P. Robertson.*

Cecchini, Gino. Sguardo alla struttura geometrico-dinamica del nostro universo. Univ. e Politecnico Torino. Rend. Sem. Mat. **8**, 33–47 (1949).
Report of a lecture.

MECHANICS

***Roy, Louis. Cours de mécanique rationnelle. Tome IV. Problèmes et exercices, suivi d'un appendice sur les fusées.** Gauthier-Villars, Paris, 1950. xi+276 pp. 1250 francs.
The first three volumes (2d ed.) were published in 1944, 1945, 1945.

Bammert, Karl, und Schmidt, Adam. Die Kinematik der mittelbaren Pleuelanlenkung in Fourier-Reihen. Ing.-Arch. **15**, 27–52 (1944).
The kinematics of the dynamic balancing of multicylinder engines has been treated successfully by resolving the motions of the parts into their Fourier series components. The cases in which the pistons are directly connected to the crankshaft by connecting-rods (unmittelbare Pleuelanlenkung) are considered in chapters XI and XII of "Technische Dynamik" by C. B. Biezeno and R. Grammel [Springer, Berlin, 1939]. In an earlier work by the first author of this paper similar mathematical treatment is said to have been applied to articulated engines (mittelbare Pleuelanlenkung) in which some of the pistons are not directly connected to the crankshaft but are joined by offset connecting-rods (Nebenpleuel) to points on the direct or principal connecting-rod (Hauptpleuel). The offset distance is an auxiliary crank. In that work, the harmonic components up to the seventh order were derived. In the present paper, the analysis is carried to the ninth order. Detailed analysis of the resolution is given and the summation formulas are derived. The parameters used are the ratio of the length of the crank to the length of its coupled connecting-rod, the ratio of the length of the auxiliary crank to the length of its coupled offset connecting-rod, the angles between cylinder axes and the fixed angles between the principal connecting-rods and the auxiliary cranks. The results are summarized in a table which gives the summation formulas for the Fourier coefficients and tables which give the Fourier coefficients in closed form up to the ninth order in terms of the foregoing parameters and from which the numerical values are easily computed.
M. Goldberg (Washington, D. C.).

Macmillan, R. H. Two aspects of rocket flight. Stabilizing finless rockets by spinning and the potentialities of multistage rocket projectiles. Aircraft Engrg. **22**, 46–47 (1950).

***Signorini, A. On some problems of rigid dynamics.** Proc. Seventh Internat. Congress Appl. Mech., 1948, v. 4, pp. 226–236.
The problems considered are the following. (I) The unrestricted problem of exterior ballistics. The author summarizes his results published elsewhere [Atti Accad. Naz. Lincei. Mem. Cl. Sci. Fis. Mat. Nat. (8) **1**, 1–41 (1946); these Rev. **9**, 109] on the reduction of the order of the equations to the lowest possible order. (II) Extension of Culmann's ellipse to the field of rigid dynamics [cf. Ann. Mat. Pura Appl. (4) **24**, 1–11 (1945); these Rev. **9**, 162]. (III) Precessional rotations of an unsymmetrical heavy top [cf. G. Grioli, Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **4**, 420–423 (1948); these Rev. **10**, 335].
D. C. Lewis (Baltimore, Md.).

Pippard, A. J. S., and Duncan, J. E. The stresses in an artillery wheel. Quart. J. Mech. Appl. Math. **2**, 398–411 (1949).

Milosavljević, Miodrag. Contribution to the analysis of suspension bridges. Glas Srpske Akad. Nauka **195**, 65–78 (1949). (Serbian)

Egerváry, Jenö. Application of Rayleigh's method to the determination of the critical speed of a rotating system. Mat. Lapok **1**, 16–26 (1949). (Hungarian. Russian and English summaries)
The author explains Rayleigh's method by applying it to the longitudinal vibrations of a system consisting of a beam clamped at one end and carrying a concentrated mass at the other end. At first the mass of the beam is neglected in comparison with the concentrated mass, and a correction of this first approximation for the lowest characteristic frequency of the system is obtained from a consideration of the kinetic and potential energies. The principal problem discussed in this paper concerns a system of importance in turbine manufacture: a rotating shaft carrying a heavy cylinder (rotor). The diameter of the shaft may be staggered, but the whole system is assumed to be symmetric with respect to a plane perpendicular to the shaft. The author writes down the partial differential equations governing the oscillations of this system, together with the transition and boundary conditions; and after neglecting the mass of the shaft, he solves (exactly) this boundary value problem. The transcendental equation obtained for the lowest characteristic frequency has only two essential parameters in it and hence a nomogram could be (but is not in the paper) constructed for its solution.
A. Erdélyi (Pasadena, Calif.).

Wuest, W. Die Biegeschwingungen einseitig eingespannter gekrümmter Stäbe und Rohre. Ing.-Arch. **17**, 265–272 (1949).
A vibration analysis is carried out for the following system: an elastic spring has its neutral-axis curved in the form of a circle; one end of the spring is built-in, while the other end supports a concentrated mass but is otherwise unrestrained. The natural, small vibrations of the system in the plane of the neutral-axis are studied. The results are of practical interest in such instances as the design of Bourdon-tube mechanisms. By conventional methods, the

sixth-order differential equation governing the motion of the system is deduced, and the characteristic equation for the (frequency) eigen-parameter is determined. After briefly examining the case of a massless spring, the behavior of the system when the spring has uniform mass and stiffness properties along its length is studied in some detail. Calculations are presented which permit rapid evaluation of the lowest six natural frequencies for a particular configuration.
M. Goland (Kansas City, Mo.).

✱**Krall, Giulio. Forced or self-excited vibrations of wires.** Proc. Seventh Internat. Congress Appl. Mech., 1948, v. 4, pp. 221–225.

This is a brief and scarcely intelligible summary of various results relating to the vibrations of a wire suspended between two points. Seemingly the work in question is the same as that reported in four notes which have been reviewed previously [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **3**, 11–17, 17–22 (1947); **5**, 197–203, 285–288 (1948); these Rev. **10**, 629]. *L. A. MacColl.*

✱**Mack, Charles E., Jr. Tensor analysis of aircraft structural vibration.** Institute of Aeronautical Sciences, 1946. 66 pp.

The tensor method of Kron is applied to the study of the vibrations of an airplane. The analysis presupposes that the continuous structure may be approximated by a system with a finite number of degrees of freedom. To calculate the kinetic energy "absolute coordinates" are used, referring the motion to a frame fixed in space. The potential energy is expressed in "relative coordinates" on which the deformation of the structure depends. The absolute coordinates are expressed in the relative coordinates and ignorable coordinates, connected with the rigid body motion. Ignoring the latter, the dynamic matrix is found. The whole airplane is split up in a number of subsystems with no static coupling between them. The kinetic and potential energy of the subsystems (e.g., wing, fuselage, tail-unit) are easier to calculate than those of the whole airplane. The composition of the subsystems is effected by a transformation of the coordinates applied to the known energies of the subsystems. The number of degrees of freedom in this final calculation can be diminished by using only a small number of the modes of the subsystems, when only a limited number of the frequencies and modes of the whole airplane must be known. At the end the author gives examples using a very highly idealized airplane. *W. H. Muller.*

✱**Mokrzycki, G. A. Calculation of the disturbed motion of an aircraft by an inversion theorem for Laplace transformation.** Proc. Seventh Internat. Congress Appl. Mech., 1948, v. 2, pp. 548–553.

Mizuno, Tadasi. Transverse vibration of a column having a longitudinally moving mass at its end. J. Soc. Appl. Mech. Japan **2**, 101–103, 110 (1949). (Japanese. English summary)

✱**Mazet, R. Application de la methode de "l'effet d'accompagnement" à la détermination de la vitesse critique de vol et du dégre d'explosivité du flutter.** Proc. Seventh Internat. Congress Appl. Mech., 1948, v. 4, pp. 96–108.

Rubbert, F. K. Zur Theorie des momentfreien Kreisels. Ann. Physik (6) **5**, 237–250 (1949).

Rigorous formulas and simple approximations for the "mean" velocities of precession and rotation of the unsymmetric gyroscope under the action of no moments are the chief results of this variation of a classical theory.
D. C. Lewis (Baltimore, Md.).

Grioli, Giuseppe. Questioni di stabilità riguardanti le precessioni regolari del solido pesante asimmetrico. Ann. Scuola Norm. Super. Pisa (3) **1** (1947), 43–74 (1949).

In a previous paper [Ann. Mat. Pura Appl. (4) **26**, 271–281 (1947); these Rev. **10**, 335], the author showed the existence of ∞^2 regular precessional motions of a heavy unsymmetrical top. The linear stability of these motions is investigated in the present paper. As four of the six characteristic exponents of the equations of variation are always zero, instability is in general the case. For bodies of very special structure, however, stability is possible. The condition for this is the consistency of four equations involving two structural parameters. Also, initial perturbations of certain special types remain small. The author makes use of a theorem, believed to be new, which gives a necessary and sufficient condition that a linear nonhomogeneous system of differential equations with periodic coefficients should have a periodic solution when it is given that the corresponding homogeneous system possesses linear first integrals with periodic coefficients. *D. C. Lewis* (Baltimore, Md.).

Woinaroski, Rudolf. Sur la stabilité des configurations d'équilibre d'un système de points matériels. Disquisit. Math. Phys. **7**, 139–149 (1948).

The author shows that the criterion for stability or instability for a system with a potential function becomes only a criterion for instability in systems without such a function. *P. Franklin* (Cambridge, Mass.).

Fumi, Fausto. Equazioni di Hamilton per sistemi olonomi a sollecitazione conservativa in senso lato. Atti Accad. Ligure **5**, 338–345 (1949).

The author considers systems of the form

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right)-\frac{\partial T}{\partial q_i}=\frac{d}{dt}\left(\frac{\partial V}{\partial \dot{q}_i}\right)-\frac{\partial V}{\partial q_i},\quad i=1,\cdots,n,$$

where $T=T(q,\dot{q})$ is the kinetic energy and $V=V(q,\dot{q})$ is a generalized potential function. The case when V is linear in the $\dot{q}$'s is especially considered. The results seem trivial to the reviewer. *D. C. Lewis* (Baltimore, Md.).

Bottema, O. On the small vibrations of nonholonomic systems. Nederl. Akad. Wetensch., Proc. **52**, 848–850 = Indagationes Math. **11**, 296–298 (1949).

It is shown that the small vibratory motions of a conservative nonholonomic system about a position of equilibrium are theoretically essentially different from the corresponding situation for holonomic systems. The well-known remark of Whittaker to the contrary seems to be due to the fact that he restricted attention to equilibrium points at which the gradient of the potential energy is zero. For nonholonomic systems it is possible to have other equilibrium points. *D. C. Lewis* (Baltimore, Md.).

Bilimović, Anton. Application of Pfaff's method to the theory of adjusted canonical variables. Glas Srpske Akad. Nauka **191**, 67–81 (1948). (Serbian. Russian summary)

In a system in which the Pfaff expression Φ for generalized coordinates q_i and impulses p_i is of the form $\Phi=\sum_{i=1}^{n} p_i q_i - Hdt$, where $H=H(p_i, q_i)$, the variables p_i, q_i are canonical because the Pfaff equations have the form of

canonical equations. If for one of the coordinates, say q_n, we have $\partial H/\partial q_n = \nu_n =$ constant, then its corresponding integral is $q_n = \nu_n t + \alpha_n$, where α_n is a new arbitrary constant. In such case q_n is the so-called "angular" coordinate and its corresponding p_n the "action" coordinate. Let us call these "adjusted" coordinates w_i and I_i, respectively. The Pfaff expression will have for them the form $\Phi = \sum_{i=1}^{n} I_i dw_i - H(I_i)dt$. Explicit expressions for I and w are derived for a conservative system with one degree of freedom. The author also considers the case of several degrees of freedom and shows how the Pfaff method can be applied to the harmonic oscillator and to Keplerian motion. In the last problem it is shown that five "adjusted" variables and the constant in the expression of the sixth are identical with Delaunay's elements of planetary motion. *L. Jacchia.*

Kasner, Edward, and De Cicco, John. Higher properties of physical systems of curves. Proc. Nat. Acad. Sci. U. S. A. **36**, 119–122 (1950).

A system S_k of ∞^3 plane curves in a field of force consists of curves along which a constrained motion is possible such that the pressure P is proportional to the normal component N of the force vector. Thus $P = kN$, where $k(\neq -1)$ is the constant factor of proportionality. In this note the authors restate two previously known properties of such systems S_k, and then state without proof some new results which are related to these properties. *L. A. MacColl.*

Hydrodynamics, Aerodynamics, Acoustics

✶Milne-Thomson, L. M. Theoretical Hydrodynamics. 2nd ed. The Macmillan Company, New York, N. Y., 1950. xxiii+600 pp. (4 plates). $8.50.

The first edition appeared in 1938. "Apart from rearrangements and new methods of presentation this edition differs from its predecessor in three important particulars: the introduction of the circle theorem [Milne-Thomson, Proc. Cambridge Philos. Soc. **36**, 246–247 (1940); these Rev. **1**, 284], whereby the disturbance of a given two-dimensional flow by the introduction of a circular cylinder can be written down without calculation; . . . the corresponding theorem for the sphere [P. Weiss, Proc. Cambridge Philos. Soc. **40**, 259–261 (1944); these Rev. **6**, 191]; the addition of a chapter on the flow of compressible fluids."

Verigin, N. N. On the rise of the level of ground water under the influence of forced infiltration. Izvestiya Akad. Nauk SSSR. Otd. Tehn. Nauk **1949**, 1723–1734 (1949). (Russian)

A mathematical model in terms of the theory of functions of complex variables is given for the study of a problem on the infiltration of water into soils. The particular problem considered is the determination of the ground water level at various points between two bodies of water, such as rivers, if forced infiltration is applied along the ridge of the water divide between the bodies of water.

H. P. Thielman (Ames, Iowa).

Verigin, N. N. On the flow of ground water with local additional infiltration. Doklady Akad. Nauk SSSR (N.S.) **70**, 777–780 (1950). (Russian)

Davydov, B. Variational principle and canonical equations for an ideal fluid. Doklady Akad. Nauk SSSR (N.S.) **69**, 165–168 (1949). (Russian)

A variational principle for an ideal fluid can be obtained by considering the displacements of the fluid particles from their positions at some initial instant as variables which may be varied independently. For the independent variables one may also choose the displacements of the particles from those virtual positions for which the density ρ of the fluid is constant. However, somewhat simpler expressions are obtained if one considers these virtual positions themselves (i.e., the Lagrangian coordinates q_k in terms of x_k and t) as variables which vary independently. The author obtains a simple variational principle by varying ρ and v_k in the Lagrangian (1) $L = \frac{1}{2}\rho v_k^2 - \epsilon(\rho)$, where $\epsilon'(\rho) = \int \rho^{-1} dp$ is the density of the potential energy. Since this cannot be done independently for various points some supplementary conditions must be imposed. If one assumes only that the equation of continuity (2) $\dot{\rho} + (\rho v_k)_k = 0$ holds, the equations for a vortex-free fluid are obtained.

In order to get the equations for a vortex fluid the author introduces instead of three Lagrangian coordinates a single coordinate α of each particle, this coordinate satisfying the condition (3) $\dot{\alpha} + \alpha_k v_k = 0$.

By requiring that (3) holds and that the values of α at the initial and terminal instants remain unchanged while variation takes place, the author fixes one coordinate of each particle at these instants. Adding the left hand sides of (2) and (3) to the Lagrangian (1), after multiplying them by undetermined multipliers $-\lambda$ and φ (considered as functions of x_k and t) respectively, and varying all quantities independently in the resulting expression, the desired equations follow in the usual way. The canonical equations for λ, α and ρ, φ are found to be identical with the Lagrangian equations obtained above. Finally there is mentioned an application to a quantum liquid. *E. Leimanis.*

Steinbuch, K. Ein Verfahren zur Lösung eindimensionaler Ausgleichsvorgänge. Ing.-Arch. **17**, 233–242 (1949).

For time $t<0$ the fluid between the planes $x=0$ and $x=x_1$ is at rest. At time $t=0$, the fluid is to be subjected to an instantaneous disturbance at the plane $x=0$. This disturbance may take the form of a sudden change in the velocity v, or in $\int v dt$, or in dv/dt, etc. On the wall $x=x_1$, the velocity is to meet one of the conditions $v=0$, $\partial v/\partial x=0$, or $\partial v/\partial x = \lambda \partial v/\partial t$. Finally, for $0<x<x_1$, $t>0$, the velocity v of the fluid is to satisfy the equation $\partial v/\partial t = a^2 \partial^2 v/\partial x^2$. Using $2\pi^{-\frac{1}{2}}\int_0^{x/(2a\sqrt{t})} e^{-y^2} dy$ as the basic function, the above problems are solved by the method of images. *F. G. Dressel.*

Pignedoli, Antonio. Sui vortici cilindrici. Atti Sem. Mat. Fis. Univ. Modena **3**, 102–124 (1949).

The author is concerned with finding plane motions of a perfect fluid subject to no extraneous force which are steady in a frame of reference rotating at uniform angular speed ω with respect to an inertial frame and which are rotational inside a certain contour and irrotational outside it. Such solutions, which represent a type of finite vortex, are known for the case when the vorticity w is constant, but the author is concerned primarily with the case when w varies. He shows that $w = w(\Psi)$, where Ψ is the stream-function for the flow relative to the rotating frame, and hence the vorticity is constant on each stream-line in this frame. He reduces the problem to the integration of $\nabla^2\Psi + w(\Psi) - 2\omega = 0$, regarding w as a given function of Ψ. He sketches a proof of existence

of solutions for the case when w is a linear function of Ψ, based upon integral equation methods, and exhibits the solution for the case of a circular boundary. The paper concludes with a solution in which the boundary is an epicycloid and the vorticity within it is uniform.

C. Truesdell (Washington, D. C.).

Strang, J. A. Superposable fluid motions. Communications Fac. Sci. Univ. Ankara 1, 1–32 (1948).

The author calls two solenoidal velocity fields v_1 and v_2 satisfying Navier's equation for the motion of a viscous incompressible fluid "superposable" if their vector sum v_1+v_2 is also a solution subject to suitable pressure and force fields. He calls a field v "self-superposable" if $2v$ is similarly a solution. He derives the condition curl $(v_1\times w_2+v_2\times w_1)=0$, where w_1 and w_2 are the respective vorticities, as necessary and sufficient for superposability, and the condition curl $(w\times v)=0$ for self-superposability. In particular, he notes that irrotational and Beltrami motions $(w\times v=0)$ are self-superposable, and that if two motions are such that the stream-lines of either coincide with the vortex-lines of the other, they are superposable. [Reviewer's note: the author's analysis of self-superposable motions could have been simplified by using the fact that any circulation-preserving motion with steady vorticity is self-superposable, and that for this class of motions of viscous fluids the main theorems of the theory of inviscid fluids remain valid.] The author notes that the classical steady motion theorems of d'Alembert (w=constant on each stream-line in plane motion) and Lamb (existence of Bernoullian surfaces) thus appear as statements that steady motions of inviscid incompressible fluids subject to conservative forces are always self-superposable. If v_1 is irrotational, a class of rotational flows superposable upon it can always be found; if curl curl $v_1=0$, a class of superposable irrotational flows can be found. The author gives special attention to plane motions, and shows that Rankine's construction of stream-lines by superposition is valid only for irrotational flows or for flows in which $w=k\psi$, where ψ is the stream function and k has the same value for each. In a self-superposable flow the vorticity is propagated in accordance with the ordinary diffusion equation $\partial w/\partial t-\nu\nabla^2 w=0$. In the remainder of the paper the author discusses special cases. He finds some new exact solutions, and states that most of those already known are self-superposable. *C. Truesdell.*

***Björgum, Oddvar. On some three-dimensional solutions of the non-linear hydrodynamical equations.** Proc. Seventh Internat. Congress Appl. Mech., 1948, v. 2, pp. 341–350.

The author constructs two families of exact solutions of Navier's equation and the continuity equation for incompressible fluids. His first result was obtained earlier by V. Trkal [Časopis Pěst. Mat. Fys. 48, 302–311 (1919)], while his second result is included in one of Berker [Sur quelques cas d'intégration des équations du mouvement d'un fluide visqueux incompressible, Taffin-Lefort, Paris-Lille, 1936, see pp. 136–137]. Nevertheless the author's method of expansion in complex Fourier series presents the results in a more concrete form than previously known and may prove valuable in the examination of simple cases.

C. Truesdell (Washington, D. C.).

Curtiss, Charles F., and Hirschfelder, Joseph O. The thermodynamics of flow systems. J. Chem. Phys. 18, 171–173 (1950).

In a treatment of matter in gross mechanical equilibrium one can define entropy and then through a statistical mechanical analysis prove the second law. In this paper the authors have done the same for nonequilibrium flow systems. Defining the entropy in exactly the same manner as the static case, they prove by the use of nonequilibrium statistical mechanics (kinetic theory) that the increase in entropy due to irreversible processes involving heat conduction, viscosity, diffusion and chemical reactions is positive or zero. This is stronger than the second law of thermodynamics which only requires that this be true of the entire system. If the mechanism of energy transfer is radiation, the stronger statement would no longer apply.

H. S. Tsien (Pasadena, Calif.).

***Abody-Anderlik, E. Friction in variable density fluid.** Proc. Seventh Internat. Congress Appl. Mech., 1948, v. 3, pp. 4–12.

The author employs rough kinetic theory arguments to derive the relation $\tau=\nu d(\rho u)/dy$ connecting viscous stress τ, kinematic viscosity ν, density ρ, and velocity u in a rectilinear shearing flow parallel to the (x, z)-plane, intended to replace the usual Newton-Maxwell friction law $\tau=\nu\rho du/dy$. An approximate solution representing diffusion of two strata of fluid of different density is constructed. [Reviewer's note. The author's proposal does not satisfy the requirement of Galilean invariance. A superposed uniform translational velocity U increases the stress by an amount $\nu U\partial\rho/\partial y$.] *C. Truesdell* (Washington, D. C.).

Truesdell, C. Bernoulli's theorem for viscous compressible fluids. Physical Rev. (2) 77, 535–536 (1950).

Voronjec, Kostantin. The motion of a fluid caused by temperature changes. Glas Srpske Akad. Nauka 195, 45–64 (1949). (Serbian)

Verschaffelt, J. E. On the dynamics of gas mixtures. Simon Stevin 27, 52–64 (1949). (Dutch)

Consider a mixture of n arbitrary gases, which satisfies, when at rest and homogeneous, the equation of state for each gas and Dalton's law of partial pressures. Disregarding temperature, gravity, and external effects, the author studies the motion of the mixture when it is not homogeneous. Defining $c=\sum_{\nu=1}^n c_\nu$ and $p=\sum_{\nu=1}^n p_\nu$, where c_ν and p_ν are the density and partial pressure of the νth gas, respectively, he shows that the velocity defined by $c(d\mathbf{u}/dt)=-\text{grad } p$ is not identical with the barycentric velocity defined by $c\mathbf{u}=\sum_{\nu=1}^n c_\nu\mathbf{u}_\nu$, this result agreeing with that of Chapman and Cowling [see The Mathematical Theory of Non-uniform Gases, Cambridge University Press, 1939; these Rev. 1, 187]. Finally he proves that the hydrostatic pressure on a moving surface element in the mixture depends on the orientation of that element. *W. J. Nemerever.*

Schweikert, G. Zur Theorie des Gasdrucks gegen eine bewegte Wand. Z. Angew. Math. Mech. 29, 289–300 (1949). (German. English and Russian summaries)

Using the Maxwellian distribution function of molecular velocity for a gas of specified density and temperature at rest, the author calculates the pressure acting on a moving wall. The result is expressed in terms of two parameters v_r and θ; v_r is the ratio of the velocity of the wall to the most probable

velocity of the molecules. When $v_r>0$, the wall is moving away from the gas and the gas is expanded; when $v_r<0$, the gas is compressed. The parameter θ is the correction due to deviation from the perfect gas law. For instance, if R is the gas constant, V the volume, and T the temperature, the gas pressure acting on a static wall is $RT/(1+\theta)V$. The author shows that when $\theta>0$, the pressure on the moving wall drops to zero at a finite positive value of v_r.

The reviewer wishes to point out that since the effect of the moving wall on the molecular velocity distribution function is neglected, the author's result is valid for two cases but not in general. (1) The gas is very rarefied so that the mean free path is much larger than the dimension of the wall, the so-called "free molecular flow." In this sense the author's result is a special case ("normal incidence") of previous investigations of such flows. (2) The other case is a piston set suddenly into motion with velocity corresponding to v_r at $t=0$. At $t<0$, the piston and gas are at rest. The calculated pressure is the pressure on the piston at $t=0$, but not correct for $t>0$. The reason that this pressure is different from the conventional hydrodynamic theory is the failure of the conventional theory at points where the usual macroscopic space velocity gradient is infinite. At such points, the basis of hydrodynamic theory, the whole Enskog-Chapman treatment of the Boltzmann equation, breaks down. *H. S. Tsien* (Pasadena, Calif.).

Grad, Harold. On the kinetic theory of rarefied gases. Comm. Pure Appl. Math. **2**, 331–407 (1949).

The classical theory of dynamics of nonuniform gases is that of Enskog, Chapman, and Burnett. The basis of the theory is, of course, the Boltzmann equation for the molecular distribution function f. The validity of the theory is limited to slowly varying flow. The author's approach is to assume that the molecular distribution function f is not far from the equilibrium Maxwellian distribution function f^0 corresponding to the local temperature and macroscopic velocity. The ratio f/f^0 can then be represented as a finite sum of Hermite polynomials in the peculiar velocity of the molecules with coefficients varying with space and time. These coefficients are essentially the various moments of f. The first moments are zero by the definition of peculiar velocity. There are six second moments and ten third moments. The second moments form the stress tensor. Equations for the time and space variations of these moments can be deduced from the Boltzmann equation. No limit is imposed upon the rate of space and time variations of these moments. Stopping at the third order Hermite polynomial, the author obtains 20 equations for the moments and the thermodynamic variables. The contracted third moment is the heat flux vector. A corresponding contraction of the differential equations for the moments then reduces the number of equations to 13. All these equations are of the first order since the derivatives in the Boltzmann equation are of the first order. Thus in the present theory the stress tensor and the heat flux vector are considered as full-fledged dependent variables on equal footing with the thermodynamic variables. Some specially simple cases are then studied by the author to compare the present theory with previous ones. After a computation of the characteristics of the system of equations for one-dimensional and two-dimensional flows, the author sets up the proper boundary conditions assuming that the diffusely re-emitted molecules are at the "temperature of the wall." [This means a certain relation between the fraction of specular reflection and the accommodation coefficient.] Finally the relation between entropy η and the quantity H in the H-theorem are computed for the author's third order approximation to f. For slowly varying flows, η is a good approximation to H. *H. S. Tsien* (Pasadena, Calif.).

Oswatitsch, Klaus. Fortschritte der Gasdynamik. Acta Physica Austriaca **3**, 1–21 (1949).

This is a very short introductory presentation of the field of gas dynamics, dealing with the essential concepts rather than with detailed solution methods. The discussion starts with the concept of Mach number, continues through the nature of stationary subsonic and supersonic flows, touches the transonic field, and goes into shock waves, nonsteady flow, and flows with viscosity, turbulence, and heat conduction. Particular attention is paid to the zone of influence concept and to a discussion of the realms of application of gas dynamics. *W. D. Hayes* (Providence, R. I.).

✱Nemenyi, P. F., and Prim, R. C. On the steady Beltrami flow of a perfect gas. Proc. Seventh Internat. Congress Appl. Mech., 1948, v. 2, pp. 300–314.

A steady inviscid rotational compressible flow that is isentropic on streamlines between shocks is a Beltrami flow proper if its velocity field satisfies $\mathbf{v}\times(\nabla\times\mathbf{v})=0$. It is a generalized Beltrami flow if $\mathbf{w}\times(\nabla\times\mathbf{w})=0$, where $\mathbf{w}=\mathbf{v}/a$, a being the limiting speed of flow. The properties of generalized Beltrami flows include the following: (1) Beltrami flows proper can be obtained from generalized Beltrami flows by choosing $a\equiv$constant; (2) a Beltrami flow proper is (i) a generalized Beltrami flow, or else (ii) the pressure is uniform, $\mathbf{v}^2=$constant, and all streamlines are straight; (3) a necessary and sufficient condition for a generalized Beltrami flow is constant stagnation pressure; (4) $(\nabla\times\mathbf{v})\cdot\nabla a=0$; (5) $\mathbf{A}/a^2=\frac{1}{2}\nabla\mathbf{w}^2$, where the acceleration $\mathbf{A}=\mathbf{v}\cdot\nabla\mathbf{v}$; (6) $\nabla\times\mathbf{A}=0$ if and only if $a=a(\mathbf{v}^2)$. Examples of two classes of generalized Beltrami flows are constructed: (1) pseudoplane generalized Beltrami flows, with $\partial\mathbf{w}/\partial z=0$; (2) helicoidal generalized Beltrami flows, in which all streamlines are coaxial helices. *J. H. Giese.*

Meyer, R. E. Focusing effects in two-dimensional, supersonic flow. Philos. Trans. Roy. Soc. London. Ser. A. **242**, 153–171 (1949).

Let the characteristic lines in a plane supersonic flow be $\alpha=$constant and $\beta=$constant, and let $h_\alpha d\alpha$ and $h_\beta d\beta$ be the length elements in the physical plane. The "focusing equations" are the set of linear equations for $\partial h_\alpha/\partial\beta$ and $\partial h_\beta/\partial\alpha$ in terms of h_α, h_β and the local Mach number. The author proceeds to a consideration of singularities of supersonic flows. These may be (a) singularities of the transformation from the characteristic (α,β)-plane to the physical plane, i.e., singularities of the Mach line pattern in the physical plane, or (b) singularities of the inverse transformation, or branch-type singularities [Craggs, Proc. Cambridge Philos. Soc. **44**, 360–379 (1948); these Rev. **10**, 640]. The focusing equations for disturbances of first order (finite discontinuities of velocity gradients) and of higher orders (ditto of higher derivatives of the velocity) are determined. It is shown that Craggs' results concerning the singularities remain valid even if discontinuities are present, provided that they are not first-order. When first-order discontinuities are present, the properties of singularities of type (a) ("limit lines") are modified considerably. Examples are presented in flow of jets from nozzles. Some of these properties cannot be deduced from a consideration of the Jacobian

of the transformation. Tollmien's argument [Z. Angew. Math. Mech. **21**, 140–152 (1941); these Rev. **3**, 283] concerning the origin of oblique shock waves in flows is reviewed, simplified, and extended to the more complicated singular cases already discussed. Finally the nature of the reflection of a first-order disturbance at a sonic line is determined. In an appendix a procedure is worked out for step-by-step integration of the characteristic equations near a branch line. It is shown that this special technique is required if the same accuracy is to be achieved as in the rest of the flow. Most of the results of this investigation can be extended to other two-dimensional cases, such as axisymmetric flow, and to non-isentropic flows, etc.

W. R. Sears (Ithaca, N. Y.).

Davies, T. V. Unsteady compressible flow in two dimensions. Proc. Roy. Soc. London. Ser. A. **199**, 468–486 (1949).

The present paper investigates an unsteady perturbation flow which is superimposed on a given two dimensional irrotational flow of a compressible nonviscous fluid. Owing to the assumed smallness of the perturbation flow in comparison with the basic flow, only the linear terms of the unsteady flow are considered. The problem, therefore, becomes a linear one. With the potential ϕ and stream function ψ of the given basic flow as independent variables, the perturbed velocity components u_1 and v_1 may be expressed as linear partial differential equations. Besides, u_1 and v_1 can be replaced by two independent variables α and β which have nice physical properties. For instance, if the perturbed flow is also irrotational, $\beta=0$; in addition, if it is incompressible, the result is a Laplace equation of α as expected.

A particular flow pattern is considered. In the case of both flows' having cylindrical symmetry, β is zero, α can be solved by separation of variables. If the Mach number is small, the perturbed flow is found, as the result of the presence of a Hankel function in the solution, to consist of diverging cylindrical waves. The general case is discussed, and the solution is expressed in terms of an infinite series of Bessel functions. The coefficients in the series can be determined from the characteristic solutions with classical procedures.

C. C. Chang (Baltimore, Md.).

Timman, R. Some remarks on the theory of near-sonic, near-parallel flow and its application to channel flow. Nationaal Luchtvaartlaboratorium, Amsterdam. Report F.53, i+14 pp. (1949).

The theory of a nearly sonic, nearly parallel two-dimensional flow can be reduced, in first approximation, to Tricomi's equation of mixed type [von Kármán, J. Math. Physics **26**, 182–190 (1947); these Rev. **9**, 217]. The present paper begins by a discussion of this reduction. The author writes the velocity components in the form $U=c(1+\delta u)$, $V=c\delta^n u$, the Cartesian coordinates in the form $x=l\delta^\mu\xi$, $y=l\delta^\nu\eta$. Here δ is a small constant parameter, c the critical speed, l some "characteristic length" and n, μ, ν are positive constants. It turns out that substituting U, V, x, y into the equations of motion and retaining only terms of lowest order in δ leads to a nontrivial system only if $\mu-\nu=-\frac{1}{3}$, $n=\frac{2}{3}$. In this case the resulting equations read (1) $-(\gamma+1)uu_\xi+v_\eta=0$, $u_\eta-v_\xi=0$, γ being the adiabatic exponent. Thus $u=\varphi_\xi$, $v=\varphi_\eta$, where φ satisfies a simple nonlinear equation. The Legendre transformation leads to a "Legendre potential" $\chi(u,v)$ such that $\xi=\chi_u$, $\eta=\chi_v$ which satisfies the Tricomi equation (2) $\chi_{uu}=u(\gamma+1)\chi_{vv}$. The author describes several particular solutions of (2) involving Bessel functions of order $\frac{1}{3}$ and Legendre functions. These are used to obtain a symmetrical channel flow such that along the axis of symmetry (x-axis) $u=\xi(1+a_1\delta^{1/3}\xi)$. In this case $\nu=0$ and l is taken as the half-width of the channel. Then u and v turn out to be rational functions of ξ and η. The flow exhibits the qualitative properties known to hold for the exact solution (existence of branch-line, triply covered portion of the hodograph plane).

L. Bers.

Wang, Chi-Teh, and Brodsky, R. F. Application of Galerkin's method to compressible fluid flow problems. J. Appl. Phys. **20**, 1255–1256 (1949).

★Teofilato, Pietro. Extension of an hydro gas dynamic symilitude to the flow with axial symmetry. Proc. Seventh Internat. Congress Appl. Mech., 1948, v. 4, pp. 19–27.

Van Tuyl, A. On the axially symmetric flow around a new family of half-bodies. Quart. Appl. Math. **7**, 399–409 (1950).

Following a paper by A. Weinstein [Quart. Appl. Math. **5**, 429–444 (1948); these Rev. **10**, 116] on the flow of a perfect incompressible fluid round a solid of revolution, using the method of sources and sinks, the author uses the results for the flow corresponding to a uniform disc-source at right angles to a uniform parallel flow from infinity. The stream curve, corresponding to zero value for ψ, Stokes' stream function for axially symmetric flow, is in the form of a profile extending to infinity and the corresponding surface of revolution is called a half-body. The strength of the sources over the disc is taken to be uniform and as the radius of the disc varies, with both the total source strength and the velocity at infinity remaining constant, the shape of the half-body varies also. As the radius of the disc tends to zero, the half-body decreases in bluntness until the form of a well known Blasius-Fuhrmann half-body is reached, while as the radius of the disc increases under the same conditions an upper value is reached at which the nose coincides with the disc. By means of elliptic integrals, the stream function and velocity components are determined and graphs are obtained by means of which the nose of any given profile can be plotted. Curves are also given which show the position and the magnitude of the maximum velocity on the surface as functions of the half-body. These points are of course points where the danger of cavitation is greatest.

R. M. Morris (Cardiff).

★Pulliam, Francis McConnell. Existence of a Two-Dimensional Potential Flow with Wake Past a Symmetric Convex Profile. Abstract of a Thesis, University of Illinois, 1947. i+3 pp.

This is a statement of results obtained in the thesis cited. The principal theorem is, "there exists a two-dimensional potential flow with wake with $\omega'(0)=0$ past any strictly convex profile symmetric with respect to the flow at infinity and with continuously changing slope." The function $\omega(s)$ is not defined. [See, however, Milne-Thomson, Theoretical Hydrodynamics, Macmillan, London, 1938, p. 308.]

E. Pinney (Berkeley, Calif.).

Bilharz, Herbert. Zur Theorie der tragenden Linie mit periodischer Zirkulation. Z. Angew. Math. Mech. **29**, 311–317 (1949). (German. English, French and Russian summaries)

The case treated is that of an infinite lifting line with circulation varying periodically along the span. The lift and

drag for sinusoidal circulation were calculated by Prandtl [Nachr. Ges. Wiss. Göttingen. Math.-Phys. Kl. **1918**, 451–477]. Here the author considers more general periodic functions and also calculates the downwash field behind the line.
W. R. Sears (Ithaca, N. Y.).

***Legras, Jean. Contribution à l'étude de l'aile portante.** Publ. Sci. Tech. Ministère de l'Air, Paris, no. 222, v+93 pp. (1949).

After two introductory chapters, the author attacks the problem of the thin sweptback wing in incompressible flow. He writes the induced velocity as the sum of a number of terms, but seems to leave the most troublesome of these unsolved, and proceeds to the circular wing. He shows that any lifting-surface problem can be decomposed into one without circulation and a pure circulation required to satisfy edge conditions. Therefore he treats first the circular wing without circulation. Two types of potential functions are found, related to Legendre polynomials, and a solution in series is proposed. Suitable potentials for the circulatory part are then derived from these, and the solution is formally complete.
W. R. Sears (Ithaca, N. Y.).

Souczek, E. Der Tragflügel in der nicht homogenen Strömung. (Ebenes Problem.) Österreich. Ing.-Arch. **3**, 396–404 (1949).

The linearized ("thin airfoil") theory of Birnbaum, Glauert, Munk, et al. [W. F. Durand, Aerodynamic Theory, Durand Reprinting Committee, Pasadena, Calif., 1943, v. 2, pp. 39–48] is extended to include variations in both magnitude and direction of the free stream velocity in the neighborhood of the airfoil. The formulation and the solution to the problem are in terms of Fourier series. Results are obtained for the lift distribution, lift and leading edge moment. In the opinion of the reviewer, the author's results are essentially contained in the earlier work [Durand, loc. cit.], since the only effect of variable free stream velocity in the formulation is to modify the effective incidence, and the previous solutions to the boundary value problem already provide for arbitrary incidence.
J. W. Miles.

Saharnyĭ, N. F. Flow without separation past a system of two airfoils of given shape. Akad. Nauk SSSR. Prikl. Mat. Meh. **13**, 445–448 (1949). (Russian)

The author considers a two-dimensional irrotational and steady flow of an incompressible fluid around two thin airfoils with parallel chords and of given shape. The method is based on the replacement of each airfoil by a substitution vortex sheet over which the vorticity varies continuously. Analytically the problem is reduced to a system of two integral equations, which is solved by an approximation method. The results are illustrated by the flow past two equal and parallel flat plates. In this case a good agreement is found with known results.
E. Leimanis.

Betz, A., und Krahn, E. Berechnung von Unterschallströmungen kompressibler Flüssigkeiten um Profile. Ing.-Arch. 17, 403–417 (1949).

Let $\mathfrak{W}$ be the velocity and ρ_∞ the density in an incompressible irrotational plane flow about a profile P with $\mathfrak{W}=\mathfrak{W}_\infty$ at ∞. Let $\mathfrak{w}$ be the velocity and ρ the density in a subsonic compressible irrotational flow about P, with $\rho=\rho_\infty$ and $\mathfrak{w}=\mathfrak{W}_\infty$ at ∞. Krahn has approximated $\mathfrak{w}$ by $\mathfrak{w}_1=\mathfrak{W}(\rho_\infty/\rho)^{\frac{1}{2}}$, where $(\rho/\rho^*)^{\gamma-1}=\frac{1}{2}(\gamma+1)-\frac{1}{2}(\gamma-1)(|\mathfrak{w}_1|/c^*)^2$, ρ^* being density at critical speed of sound c^*. For flow without circulation about a circular cylinder at Mach $M_\infty=0.3$ this yields a surface velocity distribution which agrees to within 2 per cent with Lamla's fourth approximation by the Rayleigh-Janzen method [Jahrbuch der Deutschen Luftfahrtforschung **1939**, 165 ff.; translated in Wright Field Translation **F-TS-6689-RE**]. This encourages the assumption that even for thick profiles $\mathfrak{w}'$, defined by $\mathfrak{w}_2=\mathfrak{w}-\mathfrak{w}_1=\mathfrak{w}'(\rho_\infty/\rho)^{\frac{1}{2}}$, is negligible compared to $\mathfrak{W}$.

In the present paper the differential equations for $\mathfrak{w}'$ have been linearized in accordance with this assumption to obtain

$$\text{(1)} \qquad \operatorname{div} \mathfrak{w}'=\lambda m^2|\mathfrak{W}|\mathfrak{W}\cdot\operatorname{grad}\mathfrak{W},$$

$$\text{(2)} \qquad \operatorname{curl} \mathfrak{w}'=\lambda m^2|\mathfrak{W}|\mathfrak{W}\times\operatorname{grad}\mathfrak{W},$$

where $m=M_\infty(c_\infty/c^*)(\rho_\infty/\rho^*)^{\frac{1}{2}}$, and

$$1/\lambda=(\gamma+1)\rho/\rho^*-\gamma(m|\mathfrak{W}+\mathfrak{w}'|)^2,$$

c_∞ being the speed of sound at ∞. On P the normal component of $\mathfrak{w}'=0$, and $\mathfrak{w}'=0$ at ∞. Throughout, the approximation $\lambda=\sum_0^3\lambda_j(m|\mathfrak{W}+\mathfrak{w}'|)^{2j}\sim\sum_0^3\lambda_j(m|\mathfrak{W}|)^{2j}$, with $\lambda_j=$constant, is used. For flow about a circular cylinder the authors exhibit a simple plane source and vortex distribution that generates a velocity field w' that satisfies (1), (2). At $M_\infty=0.4$ the approximate velocity distribution $\mathfrak{w}_1+\mathfrak{w}_2$ is indistinguishable from Lamla's.

To find the flow about a more general profile P, write (1), (2) in complex form (3) $\partial w'/\partial z=\frac{1}{2}\lambda(m\overline{W})^2dW/d\bar{z}$, where $z=x+iy$, $W=\mathfrak{W}_x+i\mathfrak{W}_y$, $w'=\mathfrak{w}_x'+i\mathfrak{w}_y'$, subscripts denoting components. Then $W=W(\bar{z})$ and (3) imply

$$\text{(4)} \qquad w'=\tfrac{1}{2}(dW/d\bar{z})\int_\infty^z\lambda m^2\overline{W}^2dz+U(\bar{z}),$$

where $U(\bar{z})$ is the velocity of an incompressible potential flow free of singularities, and $U=0$ at ∞. Condition (4) and the boundary condition for w' determine the normal component of U on P. Hence U is uniquely determined, likewise w'. To simplify the computation, express $U=\sum_0^3\lambda_j m^{2j+2}U_j$, with $U_j(z)$ independent of m. The foregoing procedure has also been modified to adapt it for flows with circulation, and a number of subsonic flows over elliptic cylinders and airfoils have been calculated. In these cases $|\mathfrak{w}_1+\mathfrak{w}_2|$ can be fitted very closely on P by

$$|\mathfrak{w}|=|\mathfrak{W}|(\rho_\infty/\rho)^{\frac{1}{2}}-0.44|\mathfrak{W}|^2(|\mathfrak{W}|^2-1)^2M^{4.5}$$
$$-0.4(\xi/l)(1-\xi/l)(1-2\xi/l)M^{3.5}d|W|/d(x/l),$$

where l is the airfoil's chord, x the abscissa measured along l from the leading edge, and $|\mathfrak{w}|=$maximum at $x=\xi$.
J. H. Giese (Havre de Grace, Md.).

Leray, Jean. Fluides compressibles. Application à l'aile portante d'envergure infinie de la méthode approchée de Tchapliguine. J. Math. Pures Appl. (9) **28**, 181–191 (1949).

The author extends the problem of H. S. Tsien to the case of finite circulation by the hodograph method with linear pressure-volume approximation. This same problem has previously been treated by C. C. Lin, L. Bers, A. Gelbart, P. Germain, and J. W. Craggs [see Craggs, Ministry of Supply [London], Aeronaut. Res. Council, Rep. and Memoranda no. **2273** (**9801**) (1949); these Rev. **10**, 754; and the references cited in the review (in the cited review read Gelbart instead of Gilbarg)]. It is shown essentially that, in the case of linear pressure-volume approximation, a two-dimensional irrotational flow of a compressible fluid about a body can be obtained by a point-transformation from a known incompressible flow about a similar body with the same circulation and the same condition at infinity. When the free-stream Mach number is less than 0.6 (this might

be interpreted to mean well below the critical) this transformation reduces to that of Prandtl and Glauert.

Y. H. Kuo (Ithaca, N. Y.).

***Germain, Paul. Quelques applications de la théorie des mouvements coniques à l'aérodynamique supersonique.** Proc. Seventh Internat. Congress Appl. Mech., 1948, v. 2, pp. 455–469.

By appropriate transformations of variables, the linear equation for the small perturbation velocities can be reduced to the two-dimensional Laplace equation and wave equation, for the regions inside and outside the nose Mach cone, respectively, in a case of conical flow. The author treats the inside region by means of complex variables. The particular physical problems considered are slender cones and flat cones. For the latter an experimental solution by means of electrical analogy is proposed. Finally it is shown how the flow about a rectangular or sweptback wing can be built up from conical-flow results. *W. R. Sears.*

Roumieu, Ch. Étude des régimes transitoires en aérodynamique supersonique à deux dimensions. Aperçu théorique sur le domaine transsonique. Recherche Aéronautique **1949**, no. 9, 47–54 (1949).

The author obtains a number of solutions for two-dimensional unsteady supersonic flow using the linearized theory. The basic solution for a wing whose effective angle of attack is suddenly changed is first presented completely. The principle of superposition is applied to give a method for the general case, in full detail for the undeformable wing. It is noted that the case $M=1$ presents no difficulty. The decelerating wing at constant angle of attack is investigated, giving the $-\frac{1}{2}$ power law for the dependence of pressure on the deceleration. *W. D. Hayes* (Providence, R. I.).

Beskin, Leon. Supersonic flow past airfoil tips. J. Appl. Mech. **16**, 329–345 (1949).

The cases treated here involve developable airfoil surfaces of arbitrary profile, all generators of which, including leading and trailing edges, are supersonic. Such a surface may terminate by being cut off, forming a blunt tip, and this tip may have an arbitrary direction. Thus the typical case here is the tip region of a conventional trapezoidal wing. Such cases are built up by superposition of two fundamental cases, the lifting plate and the symmetrical slender wedge. These are worked out in detail here, and the results are employed, by integration, to construct the more general cases mentioned. *W. R. Sears* (Ithaca, N. Y.).

Turner, M. J. Aerodynamic theory of oscillating sweptback wings. J. Math. Physics **28**, 280–293 (1950).

This is an extension of the work of E. Reissner [Tech. Notes Nat. Adv. Comm. Aeronaut., no. **946** (1944); no. **1194** (1947); these Rev. **7**, 227; **8**, 542], on oscillating thin finite-span wings in incompressible inviscid flow, to sweptback wings. The first case treated is that of a uniformly sheared rectangular wing, i.e., one whose planform is a parallelogram. In oblique coordinates the analysis resembles Reissner's, but slightly generalized approximations are required. It is found that Reissner's principal results might be used with minor modifications, but certain new functions would have to be tabulated. Finally, the basic integral equation for a wing consisting of two sweptback trapezoidal panels is set up but is not treated in detail.

W. R. Sears (Ithaca, N. Y.).

Miles, John W. Errata: The aerodynamic forces on an oscillating flap at supersonic speeds. J. Aeronaut. Sci. **17**, 124 (1950).

The author retracts part of his previous errata [same J. **16**, 442–443 (1949); these Rev. **10**, 755].

Miles, John W. On the reduction of unsteady supersonic flow problems to steady flow problems. J. Aeronaut. Sci. **17**, 64 (1950).

Magnaradze and Galin [Galin, Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **11**, 383–386 (1947); these Rev. **9**, 254; for a translation cf. these Rev. **11**, 273] have shown that if $\psi(x, y, z)$ is a solution to the wave equation $\square^2\phi=\phi_{yy}+\phi_{zz}-\phi_{xx}=0$, a solution of $\square^2\phi=k^2\phi$ is given by

$$\phi(x, y, z)=\psi(x, y, z)-\int_0^x \psi(\xi, y, z)\frac{\partial}{\partial\xi}J_0[k(x^2-\xi^2)^{\frac{1}{2}}]d\xi.$$

It is shown how this can be used to construct solutions for oscillating supersonic wings with supersonic trailing edges from equivalent steady-flow problems. A similar method has been proposed by Germain and Bader [cf. the following review]. *W. R. Sears* (Ithaca, N. Y.).

Germain, P., et Bader, R. Quelques remarques sur les mouvements vibratoires d'une aile en régime supersonique. Recherche Aéronautique **1949**, no. 11, 3–13 (1949).

The disturbance velocity potential for a harmonically oscillating wing flying at a supersonic Mach number M can be calculated from the equation $\varphi_{xx}-\beta^{-2}(\varphi_{yy}+\varphi_{zz})+k^2\varphi=0$. The potential is $\varphi\exp\{i\omega(t-xV/\alpha^2\beta^2)\}$, where x, y, z are the usual coordinates, t the time, $\beta^2=M^2-1$, V is the speed of flight, $\alpha=\sin^{-1}(M^{-1})$, ω the circular frequency, and $k^2=(\omega^2\sin^2\alpha)/(V^2\cos^4\alpha)$. If the wing surface is defined by $z=g(x, y)\exp(i\omega t)$, the boundary condition at the wing is approximately $(\varphi_z)_{z=0}=(Vg_x+i\omega g)\exp(i\omega xV/\alpha^2\beta^2)$ on S, where S denotes the wing surface projected onto the (x, y)-plane. The author shows how the solution can be set up by use of a fundamental solution (source). Because of the effects of regions off the tips, in general, this leads to an integral equation, which can be inverted if $k=0$, but if $k\neq0$ can be inverted only if the wing has no subsonic trailing edge (in which case the region involved does not affect the wing).

The case of "homogeneous flow" is also treated, i.e., $\varphi_n=r^nf_n(\chi, \theta)$, where $r^2=y^2+z^2$, $\theta=\tan^{-1}(z/y)$ and χ denotes $x/\beta r$. If $2u$ denotes $x^2-\beta^2r^2$, a separation of variables is achieved:

$$\varphi_n=f(u)h(\chi, \theta),$$
$$u^2f''+(3/2)uf'+[(k^2/2)u-(\lambda/4)]f=0,$$
$$(\chi^2-1)[(\chi^2-1)h_{\chi\chi}+\chi h_\chi+h_{\theta\theta}]-\lambda h=0.$$

The result is finally put into the form

$$\varphi_n=x^nF_{n+\frac{1}{2}}(2k^2u)H_n(\chi, \theta),$$

where $F_{n+\frac{1}{2}}(2k^2u)$ is given by a relatively simple infinite series and $x^nH_n(\chi, \theta)$ is the solution for a steady-flow problem $(k=0)$. It is proposed to construct practical solutions by superposition. As examples, the oscillating flat plate of infinite span and the oscillating delta wing are considered. The latter is characterized by

$$(\varphi_z)_{z=0}=\sum_{j=1}^{\infty}a_jx^{j-1}\exp\{-i\omega xV/\alpha^2\beta^2\}$$

and the solution is obtained, in a rather complicated series

form, by superposition of the solutions for homogeneous flows.

Previously, a brief report was made on the same subject [C. R. Acad. Sci. Paris **228**, 1201–1202 (1949); these Rev. **10**, 642]. *W. R. Sears* (Ithaca, N. Y.).

Moskowitz, Barry, and Moeckel, W. E. First-order theory for unsteady motion of thin wings at supersonic speeds. Tech. Notes Nat. Adv. Comm. Aeronaut., no. **2034**, 26 pp. (1950).

The determination of aerodynamic loading on thin wings executing unsteady motions in a supersonic stream is discussed, subject to the stated restriction that the second time derivative of the perturbation motion is negligible. The results are applied to the pitching and plunging motions of a swept wing with straight, supersonic, parallel leading and trailing edges and streamwise tips. In the opinion of the reviewer, the method is identical with that previously known [J. W. Miles, J. Aeronaut. Sci. **16**, 378–379 (1949); these Rev. **10**, 755], although the authors claim more generality. The essential restriction (in addition to those implied by linearization), albeit not explicitly stated either by the authors or in the reference cited, is that the trailing edge is nowhere subsonic, so that first order frequency terms associated with the wake exert no influence at the wing. *J. W. Miles* (Los Angeles, Calif.).

✱Robinson, A. On some problems of unsteady supersonic aerofoil theory. Proc. Seventh Internat. Congress Appl. Mech., 1948, v. 2, pp. 500–514.

For unsteady motion of a two-dimensional symmetrical airfoil at no incidence it is shown, by means of a source distribution, that the pressure is given by the sum of three terms. The first is the Ackeret value for steady flow, the second depends on the history over a limited time, and the third depends on the acceleration. For constant acceleration the acceleration term involves an elliptic integral, for which a practical approximation is proposed. It is found that both of the nonsteady terms are usually small compared to the Ackeret term. For supersonic speeds the results can be extended to thin airfoils at incidence. Turning to the problem of an oscillating delta wing, the author introduces a change of coordinates involving Jacobian elliptic functions. The potential equation (wave equation in three space coordinates) then takes a form for which particular solutions are $\phi(r, \zeta, \eta)=r^{-\frac{1}{2}}J_{n+\frac{1}{2}}(\lambda r)F_n^m(\zeta)E_n^m(\eta)$, where the functions on the right are a Bessel function and Lamé functions of the first and second kinds. It is proposed to construct solutions in series of these functions. [See also the author and Hunter-Tod, Coll. Aeronaut. Cranfield Rep. no. **12** (1947); these Rev. **9**, 479.] *W. R. Sears* (Ithaca, N. Y.).

Loĭcyanskiĭ, L. G. Approximate method of integration of the equations of the laminar boundary layer in an incompressible gas. Akad. Nauk SSSR. Prikl. Mat. Meh. **13**, 513–524 (1949). (Russian)

The paper presents an approximate method for solving the differential equations of two-dimensional laminar flow in the boundary layer of an incompressible gas. The method is based on an application of successive momentum equations to the assumed family of velocity profiles in the boundary layer. As is well known the main difficulty lies in the choice of the family of velocity profiles and the parameters of the family. The quantities $f=U'\delta^{**2}/\nu$, $\zeta=\tau_w\delta^{**}/(\mu U)$, $H=\delta^*/\delta^{**}$, where

$$\tau_w=\mu(\partial u/\partial y)_{y=0},\quad \delta^*=\int_0^{\infty,\delta}(1-u/U)dy,$$

$$\delta^{**}=\int_0^{\infty,\delta}u/U(1-u/U)dy,$$

are introduced as "form parameters." Assuming the velocity profiles to be independent of f, ζ, H, i.e., assuming their similarity in various sections of the layer: (1) $u/U=\varphi(\eta)$ $(\eta=y/\delta^{**})$ the equations for f, ζ and H are set up.

The simplest profile of velocities in the theory of the asymptotic boundary layer is that for the flow along a flat plate placed edgewise in the fluid stream. In this case $\varphi(\eta)$ can be calculated from the tabulated values of the ratio u/U as a function of $\xi=\frac{1}{2}y(U/\nu x)^{\frac{1}{2}}$ [cf. Toepfer, Z. Math. Physik **60**, 397–398 (1912)]. This allows one to get a sufficiently precise solution of the boundary layer equations for an arbitrary distribution of velocity U at the free stream boundary of the layer:

$$H=2.59-7.55f,\quad \zeta=0.22+1.85f-7.55f^2,$$

$$f=aU'U^{-b}\int_0^x U^{b-1}(\xi)d\xi=0.44U'U^{-5.5}\int_0^x U^{4.5}(\xi)d\xi. \tag{2}$$

The results are found to agree well with the results of the first approximation given in a previous paper of the author [C. R. (Doklady) Acad. Sci. URSS (N.S.) **35**, 227–232 (1942); these Rev. **4**, 120] and in a joint paper of Kočin and the author [ibid. **36**, 262–266 (1942); these Rev. **4**, 227]. There is a considerable divergence between the formulas of Wright and Bailey [J. Aeronaut. Sci. **6**, 485–488 (1939)] and the more precise formulas (2). This may be seen from the formulas $H=2.59$, $\zeta=0.22+4.09f$, $f=0.44U'x/U$, expressing the results of Wright and Bailey in terms of the parameters of the author.

Finally the author shows how the method can be made more precise by replacing the velocity profiles (1) by either of a one-parameter family of velocity profiles $u/U=\varphi(\eta, f)$ or a three-parameter family $u/U=\varphi(\eta, f, \zeta, H)$, the last case being not of great practical interest. *E. Leimanis.*

Timman, R. A one parameter method for the calculation of laminar boundary layers. Nationaal Luchtvaartlaboratorium, Amsterdam. Report **F. 35**, F29–F45 (1949).

In spite of the existence of two-parameter methods for the calculation of laminar boundary layers, the author seeks to improve the one-parameter method. On the basis of a discussion of the boundary conditions at infinity (with the help of the transformation used by von Mises, von Kármán and Millikan) the author assumes a velocity distribution function of the form

$$f(\eta)=1-\int_\eta^\infty e^{-\eta^2}(a+c\eta^2)d\eta-e^{-\eta^2}(b+d\eta^2).$$

In the region of accelerated flow, boundary conditions up to the third derivative of the velocity distribution function are imposed at the wall. This determines a, b, c in terms of the Pohlhausen parameter if d is set equal to zero. In the region of retarded flow the additional boundary condition of vanishing fourth derivative is introduced to determine d. According to the exact conditions on the boundary, this implies the vanishing of the shear, and hence is valid only at the point of separation. The method has been applied to a number of cases, with results usually better than the

Pohlhausen procedure, including regions of retarded flow. It still fails to reproduce the point of separation in the well-known case of the elliptic cylinder investigated by Schubauer. *C. C. Lin* (Cambridge, Mass.).

Ginzel, J. Ein Pohlhausenverfahren zur Berechnung laminarer kompressibler Grenzschichten an einer geheizten Wand. Z. Angew. Math. Mech. **29**, 321–337 (1949). (German. English, French and Russian summaries)

The author extends the Pohlhausen method to the boundary layer of a compressible fluid by including the energy equation. Polynomials of the fourth degree are used to represent velocity and temperature distributions, with different values of boundary layer thickness. The required calculation is thus the integration of a system of two ordinary differential equations. Numerical examples worked out by this method show that the skin friction is little influenced by heating and depends very little on the law of viscosity. Heat transfer depends very much on heating. The process can be easily carried over to the case of the boundary layer over axially symmetric bodies.

C. C. Lin (Cambridge, Mass.).

Yaglom, A. M. Homogeneous and isotropic turbulence in a viscous compressible fluid. Izvestiya Akad. Nauk SSSR. Ser. Geograf. Geofiz. **12**, 501–522 (1948). (Russian)

The author considers the decay of isotropic turbulence in a compressible fluid neglecting the third order correlation functions. The equations for the second order correlations are then developed and reduced to a system which determines seven such functions. One of these corresponds to that for an incompressible field, the other six to an irrotational field. Considerable discussion is given to the fact that the latter field contains acoustic waves propagating with acoustic velocity and a nonpropagating entropy wave. The Fourier transforms of the correlation functions are written down and the decay rate is determined as a function of frequency. [Since the transfer terms were omitted, all of the results could have been based as readily on a discussion of a random acoustic field in a viscous field as on a turbulent field.] *G. F. Carrier* (Providence, R. I.).

Shirogane, Zensaku. The decay of turbulence. J. Jap. Soc. Appl. Mech. **2**, 135–139 (1949). (Japanese. English summary)

In isotropic turbulence, assuming that the Reynolds stress is relaxed, that the vortex diffuses from one part of the flow to another as a random walk, and that the rate of change of the intensity of the vortex with respect to the time equals the rigidity of the fluid, we find the law of the decay of turbulence, the mean square distance through which the vortex diffuses and the diameter of the vortex.

From the author's summary.

Krzywoblocki, M. Z. On the two-dimensional steady turbulent flow of a compressible fluid far behind a solid symmetrical body. I. Momentum transfer theory. J. Franklin Inst. **247**, 33–61 (1949).

The author adapts a treatment by Schlichting of the turbulent wake behind a solid symmetrical body to take care of the effects of compressibility. He states that certain equations of other authors differ considerably from his or are incorrect. The reviewer did indeed find a few errors in the indicated references, none serious. On the other hand equations (8) and (10) of the author involve unjustified assumptions, while these equations are presented correctly in the references. The author presents an eddy stress diatensor which is unsymmetrical, this involving $\overline{u'v'}\neq\overline{v'u'}$. He then restricts himself to isotropic turbulence for which $u'=v'=w'$. *W. D. Hayes* (Providence, R. I.).

Krzywoblocki, M. Z. On the two-dimensional steady turbulent flow of a compressible fluid far behind a solid symmetrical body. II. Vorticity transfer theory. J. Franklin Inst. **247**, 137–154 (1949).

A continuation of the paper reviewed above, using a slightly different basic assumption. The conclusion is reached that the momentum and vorticity transfer theories yield different results. *W. D. Hayes* (Providence, R. I.).

Krzywoblocki, M. Z. On the boundary layer at a plane or tube in a periodically oscillating stream of compressible viscous fluid. Österreich. Ing.-Arch. **3**, 404–421 (1949).

This paper is concerned with the effect of a rigid boundary on standing wave motion of a compressible viscous fluid. In the case of incompressible viscous fluid, this problem has been studied independently by Rayleigh and Schlichting. Following the latter's procedure, the author assumes the existence of a boundary layer based on which every term of the equations of motion is estimated. Furthermore, the amplitude of the wave is assumed to be small and the equations of motion are then linearized by neglecting quadratic terms. Thus the "first-order motion" is defined by the two linear momentum equations, taking the viscosity coefficient to be constant. In doing this, the author is able to obtain a finite vertical velocity component which, according to Schlichting's result, is infinite. After the "first-order motion" is found, the "second-order motion" is calculated by iteration. In this connection, the order of magnitude of the terms retained appears to be inconsistent according to the boundary layer theory. The same problem in the case of a straight circular tube is also considered. *Y. H. Kuo.*

Lighthill, M. J. The diffraction of blast. I. Proc. Roy. Soc. London. Ser. A. **198**, 454–470 (1949).

A plane shock moves uniformly normal to a wall, and encounters a corner where the wall turns abruptly through a small angle δ. The author investigates the behavior of the shock and of the disturbed flow behind it after its encounter with the corner. He proceeds by first linearizing the problem and then by transforming to a coordinate system in which the corner and the straight part of the shock are fixed. In this system the equations are similar to those of Busemann's "conical flow" theory. In this system he determines the boundary between the disturbed and the undisturbed flows behind the shock and sets up boundary conditions. The flow equations themselves are transformed to Laplace's equation in what is essentially the pressure. After several further transformations a solution is obtained. This solution is then used to compute shock shape and pressure distribution for various shock Mach numbers. The results are given graphically. In particular, the author shows that when the corner is convex the shock diffracts normally, varying only slightly from rectilinearity. In the case of a concave corner, however, a shock "bridge" springs from the progressing shock wave back to the wall in the neighborhood of the corner, forming a configuration similar to the Mach or Y type of reflection. *D. P. Ling* (Murray Hill, N. J.).

Cabannes, Henri. Étude de la singularité au sommet d'une onde de choc attachée, dans un écoulement à deux dimensions. C. R. Acad. Sci. Paris **229**, 923–925 (1949).

This is a note, including tantalisingly interesting equations and remarks, which has been rendered almost unintelligible owing to the compression demanded of authors according to the regulations of the Comptes Rendus. The paper is concerned with the heterentropic two-dimensional flow behind a stationary shock attached to an obstacle, neglecting viscosity and heat conduction. It seems likely to the reviewer that the paper, if expanded to about ten times its length (with figures, tables, graphs, etc.), would have added considerably to our knowledge, especially relating to flow at large Mach numbers. But it is impossible to be certain of anything, even of this, when the complicated investigation is presented in two brief pages. *M. J. Lighthill.*

Kai, W. Application of the computational treatment of shock waves and vortex flow to a body composed of a cone and a cylinder. Tech. Rep. **F-TS-1207-IA (GDAM A9-T-16)**. Headquarters Air Materiel Command, Wright-Patterson Air Force Base, Dayton, Ohio. iii+32 pp. (1949).

[Translated from Technische Hochschule Dresden, Peenemunde Archiv 44/14 (1944).] The author carries out a numerical computation to determine the shock front and the flow about a body of revolution consisting of a cone followed by a cylinder. The initial shock front angle is taken from Taylor and MacColl. Thereafter it is computed and due account of its curvature taken by including a vorticity term in the hodograph equations which form the basis of the flow calculation. *D. P. Ling.*

Morduchow, Morris, and Libby, Paul A. On a complete solution of the one-dimensional flow equations of a viscous, heat-conducting, compressible gas. J. Aeronaut. Sci. **16**, 674–684, 704 (1949).

This is a detailed computation of shock waves in air assuming the validity of the Navier-Stokes equation of motion and utilizing a result of R. Becker [Z. Physik **8**, 321–362 (1922)] that if the Prandtl number is equal to 3/4, the sum of the kinetic energy and the enthalpy of the gas is a constant everywhere in the gas. This simplification allows direct quadrature of the differential equation. The author gives numerical results on velocity, pressure, entropy profiles and shock wave thicknesses for various Mach numbers and viscosity-temperature relations. The same problem is treated again by A. E. Puckett and H. J. Stewart [see the following review]. The paper is concluded with some curious solutions which do not satisfy the Rankine-Hugoniot relations of the shock wave. There is as yet no physical meaning for these solutions. *H. S. Tsien.*

Puckett, A. E., and Stewart, H. J. The thickness of a shock wave in air. Quart. Appl. Math. **7**, 457–463 (1950).

In this note the authors extend the earlier work of Becker [Z. Physik **8**, 321–362 (1922)]. They consider a one-dimensional flow of a viscous heat-conducting gas for which the specific heats, the viscosity and the heat transfer coefficients may be functions of the temperature. It turns out that if the Prandtl number alone can be considered constant and of value 0.75 (which remarkably enough is nearly the case) the solution can be given by quadratures. The main problem to which the results are applied is that of shock wave thickness. This is shown to vary from about 40 molecular mean free paths for very weak shocks to 2 or 3 for strong ones. *D. P. Ling* (Murray Hill, N. J.).

Thomas, T. Y. The distribution of singular shock directions. J. Math. Physics **28**, 153–172 (1949).

In this paper the author continues his investigations of relations existing at the vertex of a pointed symmetric obstacle in a two-dimensional supersonic flow. The oncoming flow is supposed uniform, of Mach number M; no special assumptions are made concerning the flow behind the attached shock. These relations, derived in earlier papers by the author, involve the successive derivatives of shock curvature with respect to arc length along the shock, and of obstacle curvature with respect to arc length along the obstacle, the derivatives in both cases being evaluated at the vertex. Also involved are M and α, the angle between the shock and the symmetry axis. This angle can take on values ranging from the Mach angle up to 90°. The set of relations involves a sequence of functions of M and α, the vanishing of any one of which makes the set indeterminate. Any direction α which accomplishes this for a given M is said to be singular with respect to that value of M. It suggests itself that such values of α are physically unrealizable. The author divides the permissible range of α into two parts by an angle $\beta(M)$. This is the angle at which the flow behind the shock at the vertex just becomes subsonic. He then proves that, for a given M, there are no singular directions $\alpha \leqq \beta(M)$, while for $\alpha > \beta(M)$ the singular directions are dense. This appears to have bearing on the fact that, of the two possible shock directions compatible with a given set of conditions, nature regularly chooses the smaller. *D. P. Ling* (Murray Hill, N. J.).

Havelock, T. H. The wave resistance of a cylinder started from rest. Quart. J. Mech. Appl. Math. **2**, 325–334 (1949).

Consider the wave motion caused by a circular cylinder moving with uniform velocity in an ocean of infinite depth and submerged a distance which is large compared to its diameter. Under these assumptions the usual linear theory [e.g., Lamb, Hydrodynamics, 6th ed., Cambridge University Press, 1932, pp. 410 ff.] derives the expression for the steady state wave motion, the so-called "practical" solution, either by the artifice of the Rayleigh frictional coefficient, or, in the case of the direct solution, which contains an expression for regular waves in advance as well as to the rear of the cylinder, by superposing a free infinite wave train canceling out those in front. In this paper the author considers the unsteady motion caused by the cylinder starting from rest, and obtains the practical solution for the steady state in the limit (as time becomes infinite), without recourse to either of these devices. His method, which applies also to general accelerated motions, consists in integrating over time the disturbances due to small horizontal displacements of the cylinder. The expression for the velocity potential induced by such a displacement was given by the author in an earlier work [Proc. Roy. Soc. London. Ser. A. **93**, 520–532 (1917)]. In this way, expressions are obtained in the form of integrals for the surface elevation and velocity potential as functions of time. The author derives asymptotic formulas for the elevation and the wave resistance, using Blasius's theorem for the latter. Numerical calculations and curves of the wave resistance as a function of time are discussed for three different speeds. The paper concludes with the generalization of this method to accelerated

motion of a body in three dimensions, with an application to slender ship forms. The resulting formula for the potential is equivalent to that obtained by Sretensky in another way. *D. Gilbarg* (Bloomington, Ind.).

Ursell, F. Surface waves on deep water in the presence of a submerged circular cylinder. I. Proc. Cambridge Philos. Soc. **46**, 141–152 (1950).

A method is given for calculation of the surface waves of small amplitude generated in deep water by a normal velocity distribution of period $2\pi/\sigma$ prescribed over a submerged circular cylinder. The motion is assumed two-dimensional and the water inviscid. The method consists in the expansion of the potential function in a series of multipoles at the center of the cylinder. This leads to an infinite set of linear equations for the coefficients which can be solved by infinite determinants. The method is applied to the case of a train of waves incident upon a fixed circular cylinder, and it is shown that the reflection coefficient is zero and that there is a phase change in passing over the cylinder, a result first established by W. R. Dean [same Proc. **44**, 483–491 (1948); these Rev. **10**, 165]. It is next applied to the case of a uniformly expanding and contracting circular cylinder.
J. V. Wehausen (Providence, R. I.).

Sretenskiĭ, L. N. On the waves generated by an underwater source under the surface of a sphere. Izvestiya Akad. Nauk SSSR. Ser. Geograf. Geofiz. **13**, 473–496 (1949). (Russian)

The problem considered is the generation of gravity waves by a source of oscillating strength within a layer of water covering a solid sphere, the layer being of uniform thickness when at rest. The usual linearized free boundary condition is used but the shallow water approximation is not made. No rotation of the sphere is assumed. The method followed is to write the velocity potential as a source potential plus a remainder expanded in a series of Legendre polynomials. The bulk of the paper is devoted to a detailed study of the coefficients in the series with the ultimate aim of obtaining asymptotic formulas for the shape of the free surface. For the case when the solid core vanishes and the frequency of oscillation is high, an asymptotic formula for the free surface is given. This result is then extended to the case when the output of the source is approximately a single square pulse of short duration. The next case considered is that when the layer of water is very thin compared to the radius of the core and the source is at the bottom of the water.
J. V. Wehausen (Providence, R. I.).

Peters, A. S. A new treatment of the ship wave problem. Comm. Pure Appl. Math. **2**, 123–148 (1949).

The author considers in both two and three dimensions the wave motion generated on the surface of a heavy inviscid liquid by the uniform, rectilinear motion of a concentrated region of pressure (δ-function). Although a moving pressure point is in most respects a poor approximation to a ship, the wave patterns present the same gross structure. This problem, originally treated by Kelvin, is considered at some length in Lamb's Hydrodynamics and has been elaborated by Havelock and Hogner, the latter extending the method to obtain the wave resistance of more realistic ship forms. The essential novelty of this paper is a more exact treatment of the form of the free surface. It is shown that on the line traversed by the moving point the amplitude is finite and decreases as $x^{-\frac{1}{2}}$ where x is the distance back from the point, whereas previous treatments had shown infinite amplitudes. The asymptotic form at large distances for the rest of the surface is also determined more exactly than heretofore by use of the method of steepest descent. As the author points out, his potential function is equivalent to one implied by Lamb's treatment even though Lamb makes no use of it. However, the derivation of the potential function is quite different from Lamb's and is carried out more rigorously. *J. V. Wehausen* (Providence, R. I.).

Dressler, Robert F. Mathematical solution of the problem of roll-waves in inclined open channels. Comm. Pure Appl. Math. **2**, 149–194 (1949).

The author's summary is as follows. The purpose of this paper is to obtain solutions which are periodic with respect to distance, describing the phenomenon called "roll-waves," for water flow along a wide inclined channel, and to discuss the behavior of the mathematical solutions. The basic idea presented in part I is that discontinuous periodic solutions can be constructed by joining together sections of a continuous solution through shocks (or "bores"). It is shown first that no continuous solutions can be periodic and that only one special continuous solution can be used as the basis for constructing discontinuous periodic solutions. The analysis is based upon the nonlinear partial differential equations of the "shallow water theory," augmented by the Chezy formula to allow for turbulent resistance. The Bresse profile equation is obtained in a form applicable for progressing wave flows. Shock conditions are derived for the case of an arbitrary continuous channel bed and for a flow subject to a resisting force. The special continuous solution is explicitly obtained and analyzed. Branches of it are then joined together through shocks. It is proved that roll-waves cannot occur either if the resistance is zero or if the resistance exceeds a certain critical value. As the resistance decreases, the size of the waves decreases also; and if the resistance becomes too large, the profiles reverse their direction and can no longer be joined by shocks. This critical value is reached when the (dimensionless) resistance coefficient equals one-fourth the value of the channel slope. The presence of a resistance force which varies merely with velocity is not sufficient to permit the construction of periodic solutions; the resistance must also act in such a manner that it decreases as the water depth increases. The analysis proves that the ratio of wave height to wave length of roll-waves is always independent of the speed of the waves. Explicit expressions for water height and shock height as functions of wave length are derived. The investigation studies the static discharge rate as a function of the wave speed, and asymptotic formulas for the wave speed in terms of the average discharge rate are derived. Twelve sets of curves are presented, based on the equations obtained here, to illustrate the quantitative behavior of roll-waves; these may be used to check this theory against observed data. For prescribed values of slope, resistance, and wave speed, there is a one-parameter family of roll-wave solutions. If the wave length is also prescribed, the solution will then be unique.

The purpose of part II is to demonstrate the possibility of obtaining roll-wave solutions as continuous functions. Equations representing higher approximations to the basic equations, obtainable by a perturbation method due to Friedrichs, are studied for flow down a slope. Resistance effects are considered in the lowest approximation, but are neglected in the higher approximations. Transformation is made to coordinates moving down the slope in order to

solve for progressing waves. The only solution which can yield periodic waves in the perturbation is the constant "critical" flow. The nonlinear second order equation obtained for the wave profile is of the same basic type as the one first obtained by Korteweg and deVries [Philos. Mag. (5) **39**, 422–443 (1895)] for flow over a horizontal bottom. Solutions are given by the Jacobi elliptic function cn. Thus roll-waves are approximated by continuous "cnoidal" waves. The paper concludes with a brief discussion of the resistance problem from the standpoint of turbulence theory, using the concept of eddy-viscosity, due originally to Boussinesq, and the more general equations derived later by Rayleigh.

H. S. Tsien (Pasadena, Calif.).

✱Brun, Edmond, et Vasseur, Marcel. La mécanique des suspensions dans le cas d'un cyclone. Proc. Seventh Internat. Congress Appl. Mech., 1948, v. 2, pp. 264–279.

The authors study the motion of small particles in a two-dimensional vortex flow around a sink. This flow pattern is characterized by a constant angle between the local velocity and the local radius-vector from the sink as center. The flow lines are logarithmic spirals. The particles carried by this flow are considered to be small if the local flow pattern around them is always laminar. Outside forces on both fluid and particles are neglected. The basic differential equation leads to a particular solution with concentric circular paths which is stable, and real for particle densities above a limiting value, which depends on the flow condition. All other trajectories run tangentially into this particular path in such a way that all particles have the tendency to accumulate at their particular distance from the center. The application of this peculiar flow pattern to sediment separators is suggested. *H. A. Einstein* (Berkeley, Calif.).

Goldsbrough, G. R. The tides in oceans on a rotating globe. V. Proc. Roy. Soc. London. Ser. A. **200**, 191–200 (1950).

In the case of the luni-solar tide K_2 in an ocean bounded by two meridians and whose depth is proportional to $\sin^2 \theta$, where θ is the colatitude, the author integrates the differential equations in a double series of orthogonal polynomials, each of which satisfies the boundary conditions. This series converges in the mean to the required solution. On account of the orthogonal property the solution can be found in explicit literal form. The method is probably of more general application. In the present instance it leads to a means of determining the critical depths at which resonance in the luni-solar tide occurs.

L. M. Milne-Thomson (Greenwich).

Haurwitz, B. Internal waves of tidal character. Trans. Amer. Geophys. Union **31**, 47–52 (1950).

The author derives periods of internal tidal waves by perturbation theory, neglecting the earth's curvature, but including the earth's rotation. It is shown that the earth's rotation reduces the free periods at the boundaries so that resonance between them and the tidal forces is theoretically possible. Thus the previously unexplained observations of large amplitude internal waves of tidal period are explained. *H. Panofsky* (New York, N. Y.).

Rombakis, Sokrates. Über ein Integral der nichtlinearen hydrodynamischen Gleichungen und seine Anwendung in der Meteorologie. Z. Meteorologie **2**, 241–244 (1948).

The vorticity equation derived from the equations of motion and the equation of continuity in two-dimensional, nondivergent and incompressible flow, can be written as a 4th order nonlinear differential equation for the stream function. If the vorticity depends on the stream function only the nonlinear components cancel, and a solution can be found in terms of Legendre polynomials for the stream function of waves of finite amplitudes. For both the northern and southern hemispheres, the periods of some of the spherical harmonics of large n agree well with rainfall periods found empirically by Defant. *H. Panofsky.*

Hardtwig, E. Zur formalen Theorie des Gradientwindes. Z. Meteorologie **2**, 308–313 (1948).

The gradient wind equation is again derived from the equation of motion. Special cases with circular isobars are treated which show that anticyclonic circulations around low pressure systems can exist in theory and that inertia motion is possible with cyclonic motion. *H. Panofsky.*

Märbt, H. Beiträge zum Leewellenproblem. Z. Meteorologie **2**, 330–334 (1948).

It is shown by use of the Bernoulli equation that the wind over a plateau may vary periodically with height, in agreement with observations of meteor trails. A similar result had been obtained previously only in a more complicated manner. It is shown that the rotation of the earth is unimportant for this problem, provided heights of less than 100 km are considered. *H. Panofsky.*

Schmitz, H. P. Über die Änderung der Turbulenzenergie. Z. Meteorologie **2**, 338–339 (1948).

An expression is derived for dissipation of turbulent kinetic energy, greatly resembling Richardson's equation, but with a more general term for the work done against gravity. *H. Panofsky* (New York, N. Y.).

Gião, Antonio. A new dynamical climatology: its aim and method. Geofis. Pura Appl. **15**, 114–129 (1949).

Dynamic climatology is a new branch of meteorology dealing with the average properties of perturbations superimposed on the given fields of pressure, temperature and wind. The general theory is applied to specific situations and apparently explains such observations as the decay of cyclonic storms when they approach west coasts of continents, and the motion of cyclones parallel to the isotherms. This study suggests the drawing of additional climatological charts which characterize the mean behavior of cyclones in the atmosphere. *H. Panofsky* (New York, N. Y.).

Chester, W. The propagation of sound waves in an open-ended channel. Philos. Mag. (7) **41**, 11–33 (1950).

Levine and Schwinger studied the radiation and transmission properties of a semi-infinite circular acoustical duct under certain conditions of excitation [Physical Rev. (2) **73**, 383–406 (1948); these Rev. **9**, 393]. It was found that the problem could be formulated as an integral equation of the Wiener-Hopf type and for this case could be solved. From the solution, the pertinent physical parameters could be found, that is, the radiation pattern and the reflection coefficient. Inspired by this work, the author does the problem of a semi-infinite pair of parallel plates, excited by the dominant acoustical mode. It is found that the problem may still be formulated as an integral equation of the Wiener-Hopf type and all pertinent physical parameters obtained. [Reviewer's comments. This acoustical problem is equivalent to a problem in electromagnetic excitation of the so-called E plane polarization and has been done by L. A.

Vaĭnsteĭn [Izvestiya Akad. Nauk SSSR. Ser. Fiz. 12, 144–165 (1948); these Rev. 10, 659]. The H plane polarization has been done by A. E. Heins [Quart. Appl. Math. 6, 157–166, 215–220 (1948); these Rev. 10, 89, 222] and L. A. Vaĭnsteĭn [loc. cit.].]

A. E. Heins (Pittsburgh, Pa.).

Pachner, Jaroslav. Pressure distribution in the acoustical field excited by a vibrating plate. J. Acoust. Soc. Amer. 21, 617–625 (1949).

The author calculates the distant acoustic field due to a circular plate, vibrating in an infinite rigid baffle. He bases his study on the theory of forced vibrations of plates clamped at the edge. In practice, owing to resonance and interference phenomena, only a few eigenvibrations are important for a given frequency of excitation of the plate. The effect of radial and circular nodal lines is discussed. A great number of references to earlier work are included.

C. J. Bouwkamp (Eindhoven).

Levine, Harold. Variational principles in acoustic diffraction theory. J. Acoust. Soc. Amer. 22, 48–55 (1950).

A plane wave is incident upon an infinite plane screen of negligible thickness with an aperture. The author investigates this diffraction problem theoretically by giving variational principles for the amplitude of the diffracted spherical wave at large distances from the aperture. Specifically, the author is concerned with the solution of the partial differential equation $(\nabla^2+k^2)\psi=0$ at all points in space, subject to the vanishing of the normal derivative of ψ on the screen. This problem is formulated as an integral equation from which the desired variational principles may be derived. Two variational principles are given, one of which depends on small variations of the normal derivative of ψ in the aperture and the other of which depends on small variations in the discontinuity of ψ on the screen. Particular attention is given to the plane wave transmission cross section (which may be derived from the amplitude of the diffracted amplitude). The high and low frequency behaviour of the various forms of the cross section, including comparison with the Kirchhoff theory, are studied. A comparison of the two variational principles is also given. *A. E. Heins.*

Sollfrey, William. The variational solution of scattering problems. New York University, Washington Square College, Mathematics Research Group, Research Rep. No. **EM-11.** i+29 pp. (1949).

The author presents a variational approach for obtaining the back scattering cross-section of obstacles of arbitrary shapes. The differential equation and the appropriate boundary conditions are cast into an integral equation through the use of Green's theorem. From this integral equation the desired physical parameter may be cast into a variational principle. Using suitable trial functions in this procedure one can obtain approximations to the value of the back scattering cross-section of the obstacle. First order computations for sound and electromagnetic waves incident on an obstacle are made for purposes of illustration. The variational principles employed were used by Schwinger in connection with many problems in electromagnetic theory and dynamics of continuous media [cf. H. Levine and J. S. Schwinger, Physical Rev. (2) 74, 958–974 (1948); these Rev. 10, 221; and the preceding review]. *A. E. Heins.*

Leitner, Alfred. Notes on diffraction by a circular disk. New York University, Washington Square College, Mathematics Research Group, Research Rep. No. **EM-12.** i+36 pp. (1949).

This report investigates the diffraction of a plane sound wave from a rigid circular disc of zero thickness. The recent exact calculations of C. J. Bouwkamp [Dissertation, Groningen, 1941; these Rev. 8, 179] and R. D. Spence [J. Acoust. Soc. Amer. 20, 380–386 (1948); these Rev. 10, 166] are extended to compute and discuss values for the diffracted fields near the disc and far from it. The Kirchhoff approximation is compared to the exact solution and it is found to give more favorable results for the class of configurations treated here. The reason for this state of affairs is discussed in some detail. *A. E. Heins* (Pittsburgh, Pa.).

Elasticity, Plasticity

De Donder, Th., et van den Dungen, F. H. Sur les principes variationnels des milieux continus. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 35, 841–846 (1949).

The authors formulate a variational principle for the general theory of elasticity, in which a strain energy depending only upon the displacement gradients exists. The method and the resulting formulae for the stress are those of Kirchhoff [S.-B. Akad. Wiss. (Wien) Math.-Nat. Cl. 9, 762–773 (1852)]. While Kirchhoff treated only the case when the strain energy is a quadratic function, this restriction was not used in his analysis, which was restated in full generality by Hellinger [Encyklopädie Math. Wiss., vol. 4.4 (IV 30), pp. 601–694 (1914), see § 7a]. The authors remark that a special case of their result remains valid in certain theories of plasticity. *C. Truesdell.*

Colonnetti, G. Elastic equilibrium in the presence of permanent set. Quart. Appl. Math. 7, 353–362 (1950).

The author considers the relationship between stress and strain in tension and compression for a composite system of elements loaded in parallel. Each element has an elastic limit above which plastic flow at constant stress occurs. Work hardening and a Bauschinger effect are represented by this mathematical model which is used to represent actual materials. The addition of linear creep is also considered. For a given distribution of permanent strain, a variational principle is developed for the stress in a body loaded by external forces, which is independent of the source of the permanent strain. It is based on the fact that although the permanent strain may not be compatible with a displacement function, the total strain (permanent plus elastic) is. The importance of utilizing permanent strain in the design of statically indeterminate structures is emphasized. *E. H. Lee* (Providence, R. I.).

***Brandenberger, Henry. A new theory of elasticity and strength.** Proc. Seventh Internat. Congress Appl. Mech., 1948, v. 1, pp. 14–27.

The author claims that ordinary elasticity theory "by its first mathematical equation, loses all connection between stress and strain," but he adopts the equations of that theory in the familiar form of proportionality of stress and strain traces and proportionality of stress and strain deviators. He claims (1) that a body never yields under volume changes alone, and (2) that a body yields if any one com-

ponent of its stress deviator exceeds a certain yield limit, the yield then taking place only in the direction of that component. He thus proposes that each component of the deviator be treated independently, and the results superposed, but the reviewer has been unable to find in the paper a statement of the author's stress-strain relations when the yield is exceeded. The reviewer notes that (1) contradicts the common experience that a body ruptures under sufficient hydrostatic tension, and that (2) is not an invariant concept and thus is not admissible for isotropic bodies, although it is possible that the author's vague statement should be interpreted as the Galileo-Navier condition of maximum principal stress, or as a condition of maximum intensity of the stress deviator. The author explains hysteresis and failure. As is customary in the literature of strength of materials, experimental evidence fully confirming the theory is presented. *C. Truesdell.*

Isaacs, Rufus. Planar elasticity as a potential theory. Univ. Nac. Tucumán. Revista A. 6, 263–272 (1948).

Muschelišvili's [C. R. (Doklady) Acad. Sci. URSS (N.S.) 4 (1934 III), 7–11] formulation of the general Airy solution of the problem of plane elasticity for a simply-connected domain [cf. also W. Prager, Rev. Math. Union Interbalkan. 3, 63–65 (1941); these Rev. 3, 29] is derived by means of Kasner's theory of polygenic functions [Proc. Nat. Acad. Sci. U. S. A. 14, 75–82 (1928)]. In particular, the Mohr circle appears as the Kasner circle for the polygenic stress function. The method is almost identical with the application of Burgatti's theory of analytic functions of order 2 recently given by Aymerich [Rend. Sem. Fac. Sci. Univ. Cagliari 17 (1947), 1–12 (1948); these Rev. 10, 534]. *C. Truesdell* (Washington, D. C.).

Süray, S. Sur des familles de courbes attachées aux corps élastiques ou plastiques plans. Communications Fac. Sci. Univ. Ankara 1, 33–40 (1948).

The author discusses those two-dimensional problems of elasticity and plasticity for which the lines of principal stress form isothermal nets. In the elastic case this question was first discussed by Neményi [Z. Angew. Math. Mech. 13, 364–366 (1933)]; in the plastic case, by Carathéodory and Schmidt [Z. Angew. Math. Mech. 3, 468–475 (1923)]. The method used here differs from the methods used in these earlier papers in so far as it permits discussion of the elastic and plastic problems along analogous lines. *W. Prager.*

Herpin, André. Extension des relations de Cauchy aux coefficients d'élasticité du troisième ordre. C. R. Acad. Sci. Paris 229, 921–922 (1949).

The author's problem is to obtain theoretical formulae for the three elastic coefficients of third order A, B, C in the expression for the strain energy of an elastically isotropic body as a power series in the Lagrangian strain invariants, as defined by L. Brillouin [Les tenseurs en mécanique et en élasticité, Masson, Paris, 1938, p. 234]. He calculates that according to Cauchy's molecular theory (which gives $\lambda=\mu$ for the second order coefficients) the third order coefficients satisfy $16A/11=16B/3=C$, and he states that in general there are corresponding relations expressing all the nth order coefficients in terms of any one. By comparing the theoretical formula for the coefficient of dilatation with experimental values he obtains one more relation between the elastic coefficients, whence follows $A=-.18\lambda$, $B=-.05\lambda$, $C=-.26\lambda$, in approximate agreement with values proposed by L. Brillouin [op. cit., p. 339]. *C. Truesdell.*

***Brodeau, André. Anisotropie, symétrie, hétérogénéité en élasticité.** Publ. Sci. Tech. Ministère de l'Air, Paris, no. 229, vii+61 pp. (1949).

The author shows that for isotropic and some types of aelotropic elastic bodies the assumption that a displacement discontinuity across a surface where the elastic moduli are discontinuous is transversal leads to the conclusion that the tangential stresses are zero. He regards this result as evidence that a junction of two physical bodies may not be visualized as a surface of discontinuity. He discusses also the case when there is a narrow layer in which the elastic properties vary continuously but rapidly. *C. Truesdell.*

Maccaferri, Luisa. La formula per la variazione di volume nell'elasticità ereditaria. Atti Sem. Mat. Fis. Univ. Modena 3, 46–49 (1949).

In Volterra's accumulative theory of elasticity the stress $t^i{}_j(t)$ and the infinitesimal strain $e^i{}_j(t)$ at time t are related by

$$(*)\quad t^i{}_j(t)=\lambda\delta^i{}_je^k{}_k(t)+2\mu e^i{}_j(t)+\int_0^t[\phi(t,\tau)\delta^i{}_je^k{}_k(\tau)+2\psi(t,\tau)e^i{}_j(\tau)]d\tau.$$

By solving a Volterra integral equation and employing Green's theorem the author shows that the volume change Δv for a body V bounded by a surface S and subject to extraneous force F and surface force R is given by

$$(3\lambda+2\mu)\Delta v=K(t)+\int_0^t k(t,d\tau)K(\tau)d\tau,$$

where

$$K(\tau)=\int_V \rho F(\tau)\cdot r\,dV+\oint_S R(\tau)\cdot r\,dS,$$

$k(t,\tau)$ is the resolvent kernel for $3\phi+2\psi$, r is the radius vector with respect to any origin, and the volume and surface integrals are to be carried out over the body at time t, whatever the value of τ. *C. Truesdell.*

Graffi, Dario. Sulla teoria delle oscillazioni libere in un sistema soggetto a forze elastiche con ereditarieta. Atti Sem. Mat. Fis. Univ. Modena 3, 227–247 (1949).

The author generalizes some recent work of E. Volterra [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 2, 42–47, 178–180 (1947); these Rev. 8, 546] concerning the oscillations of a system governed by V. Volterra's theory [see the preceding review]. The main result is that, subject to the assumptions that the kernel be a finite sum of exponentials and that certain quantities be small, the presence of the accumulation terms does not appreciably alter the frequencies as predicted from ordinary elasticity theory, but increases the damping factors. *C. Truesdell.*

Covezzoli, Paolina. Sulle oscillazioni forzate di una trave elastica in regime ereditario. Atti Sem. Mat. Fis. Univ. Modena 3, 261–264 (1949).

The author takes up the problem of a beam vibrating in accord with V. Volterra's theory, as treated recently by E. Volterra [see the two preceding reviews]. Using a method indicated by Graffi [Nuovo Cimento (8) 5, 53–71 (1928)], she obtains a general formula for the change of amplitude and phase in a single harmonic component in terms of the Fourier transform of the kernel. E. Volterra's result for the case when the kernel is a sum of exponentials is included as a special case. *C. Truesdell* (Washington, D. C.).

✱Picone, Mauro. Sur le calcul de la déformation d'un solide élastique encastré. Proc. Seventh Internat. Congress Appl. Mech., 1948, v. 1, pp. 41–48.

✱Swainger, K. H. Severe deformations. Proc. Seventh Internat. Congress Appl. Mech., 1948, v. 1, pp. 49–60.

✱Broglio, Luigi. A method of "equivalence" applied to the solution of problems of elasticity and of mathematical physics. Proc. Seventh Internat. Congress Appl. Mech., 1948, v. 1, pp. 98–107.

✱Signorini, A. On finite deformations of an elastic solid. Proc. Seventh Internat. Congress Appl. Mech., 1948, v. 4, pp. 237–247.

Il'yušin, A. A. Some fundamental problems of the theory of plasticity. Izvestiya Akad. Nauk SSSR. Otd. Tehn. Nauk **1949**, 1753–1773 (1949). (Russian)

In this address the author surveys past Russian work in the mathematical theory of plasticity, forecasts future trends, and takes issue with physicists and metallurgists who have criticized this theory as ignoring the known structure of metals. With regard to such criticism, the author emphasizes the desirability of a physical theory of plasticity but states that engineers will have to use the present phenomenological approach at least until such a theory is developed. He also points out the possibility that even when a physical theory of the plastic deformation of polycrystalline metals is developed, it may turn out that it is far too complex to lend itself to the analysis of stresses and strains in machine parts or structural members of complicated shapes. From the author's survey of past Russian work, it appears that the theory of small elastic-plastic deformations, a "finite" (as opposed to "incremental") theory, is definitely established in Russia as the appropriate tool of stress analysis in problems of contained plastic deformation (as opposed to problems of unrestricted plastic flow). The author indicates the well-known limitations of this theory but states, without giving details, that its predictions have never been found to involve errors of more than 7 to 10%. *W. Prager* (Providence, R. I.).

Arutyunyan, N. H. The theory of an elastic stressed state of concrete, taking account of creep. Akad. Nauk SSSR. Prikl. Mat. Meh. **13**, 609–622 (1949). (Russian)

The author discusses the deformations of an elastic continuum in the presence of creep assuming a linear relation between stresses and the amount of creep they produce. *W. Prager* (Providence, R. I.).

Sokolovskiĭ, V. V. One-dimensional nonstationary motion of a viscous-plastic medium. Akad. Nauk SSSR. Prikl. Mat. Meh. **13**, 623–632 (1949). (Russian)

Plane shear waves (displacement component $u_y=u_y(x,t)$, $u_x=u_z=0$) and cylindrical shear waves (displacement component $u_\theta=u_\theta(r,t)$, $u_r=u_z=0$) are considered in a viscous plastic medium. The relation between shear stress τ and shear strain γ is

$$d\gamma/dt=0,\quad |\tau|\leqq k;\qquad d\gamma/dt=(\kappa/\mu)(|\tau|-k),\quad |\tau|\geqq k,$$

where $\kappa=\operatorname{sgn}\tau$ and k and μ are constants. Plane shear waves determine the diffusion equation, and solutions are given for the cases of constant stress and constant velocity applied for a period to the surface $x=0$ of a semi-infinite body, after which the boundary is freed. A pair of linear simultaneous partial differential equations are obtained for cylindrical shear waves, and the solution is given for a sudden rise in shear stress at the surface, $r=r_0$, of a cylindrical hole in an infinite body, followed by linear reduction of surface stress with time. *E. H. Lee* (Providence, R. I.).

Sokolovskiĭ, V. V. Some problems of the theory of plasticity for a power-hardening material. Akad. Nauk SSSR. Prikl. Mat. Meh. **13**, 655–658 (1949). (Russian)

The strain-hardening material considered in this paper obeys a stress-strain law of the deformation type, the octahedral shearing stress being proportional to a power of the octahedral shear strain (hence the term "power-hardening"). The following two problems are treated: the bending of a wedge by a force applied to its edge and the torsion of a tapering shaft of circular cross section. *W. Prager.*

✱Hill, R. Some special problems of indentation and compression in plasticity. Proc. Seventh Internat. Congress Appl. Mech., 1948, v. 1, 365–377.

Problems of large deformation in plane strain of an ideal plastic material are considered. The main types of boundary-value problems encountered are discussed. As examples the expansion of a semi-circular hole in the surface of a semi-infinite medium, and the compression of the apex of a wedge by a smooth flat plate are treated. Stress and strain distributions are considered for both problems. *E. H. Lee* (Providence, R. I.).

Stowell, Elbridge Z. Compressive strength of flanges. Tech. Notes Nat. Adv. Comm. Aeronaut., no. **2020**, 42 pp. (1950).

A deformation theory of plasticity is combined with an analysis of finite deflections to obtain the maximum load a flange can carry before complete collapse. The differential equation for angle of twist φ is

$$Ad\varphi/dx-Bd^3\varphi/dx^3+C(d\varphi/dx)^3=0,$$

where A, B, and C are constants which include both the secant and elastic modulus. Good agreement is obtained between computed and measured values of load, strain, and deformation for cruciform sections which buckled by twisting. *D. C. Drucker* (Providence, R. I.).

Lubahn, J. D., and Sachs, G. Bending of an ideal plastic metal. Trans. A.S.M.E. **72**, 201–208 (1950).

Taking into account the induced curvature and the progressive changes in the cross-sectional shape, the problem of fully plastic pure bending of a rectangular beam is solved for the very wide beam (plane strain) and the very narrow beam (plane stress). The material obeys an incremental or flow law with the von Mises yield condition, and does not work harden. Plots are given of stress and strain distributions and the varying location of the neutral axis obtained by successive approximations. *D. C. Drucker.*

Šapiro, G. S. On the integration by quadratures of the equations of the plane one-dimensional problem of the theory of plasticity taking account of the hardening of the material. Akad. Nauk SSSR. Prikl. Mat. Meh. **13**, 659–662 (1949). (Russian)

The paper is concerned with axially symmetric problems of plane stress or plane strain. Assuming that the incompressible strain-hardening material obeys a stress-strain law of the deformation type, the author shows that the solution of problems under consideration can be reduced to quadratures. *W. Prager* (Providence, R. I.).

***Erim, Kerim. Sur le principe de Saint-Venant.** Proc. Seventh Internat. Congress Appl. Mech., 1948, v. 1, pp. 28–32.

Following R. von Mises [Bull. Amer. Math. Soc. **51**, 555–562 (1945); these Rev. **7**, 40] the author takes the two-dimensional analogue of Boussinesq's problem, originally due to Flamant [C. R. Acad. Sci. Paris **114**, 1465–1468 (1892)], and shows that Saint-Venant's principle of statically equivalent loads holds good when the forces are normal to the straight boundary, but that when the forces have tangential components they must satisfy an additional condition, which is contained in von Mises' general results.

B. R. Seth (Ames, Iowa).

Benscoter, S. U. Analysis of a single stiffener on an infinite sheet. J. Appl. Mech. **16**, 242–246 (1949).

The problem of an axially loaded straight elastic rod which is attached along its axis to a flat elastic sheet is considered as a boundary value problem in plane stress. By means of appropriate singular solutions of the equations of the theory of plane stress this boundary value problem is reduced to an integro-differential equation. It is found that this equation is of the same form as the well-known lifting-line integral equation in aerodynamics. On the basis of this fact numerical solutions are obtained by means of the so-called Multhopp procedure for the approximate solution of this equation. Symmetrical as well as anti-symmetrical loading conditions of the rod are investigated.

E. Reissner (Cambridge, Mass.).

Green, A. E., and Sneddon, I. N. The distribution of stress in the neighbourhood of a flat elliptical crack in an elastic solid. Proc. Cambridge Philos. Soc. **46**, 159–163 (1950).

A solution is given for the stress produced by a plane crack of elliptical shape in an infinite body under tension at infinity perpendicular to the plane of the crack. As the authors state, this is a very special case of the problem of the ellipsoidal cavity solved by Sadowsky and Sternberg [J. Appl. Mech. **16**, 149–157 (1949); these Rev. **10**, 760]. The method used is simpler [A. E. Green, Proc. Roy. Soc. London. Ser. A. **195**, 533–552 (1949); these Rev. **11**, 286] and takes advantage of the known solution for a flat elliptical disk moving perpendicular to its plane through an incompressible fluid. However, in this special case the interesting stresses are so highly localized and reach infinite values that the solution does not have great practical importance. Using the same method and the assumption of zero friction, the problem of the indentation of the flat surface of a semi-infinite body by a flat-ended cylindrical punch of elliptic cross-section is solved for the contact stress.

D. C. Drucker (Providence, R. I.).

Galin, L. A. On the existence of a solution of the elastic-plastic problem of torsion of prismatic bars. Akad. Nauk SSSR. Prikl. Mat. Meh. **13**, 650–654 (1949). (Russian)

The problem of elastic-plastic torsion of a cylindrical bar can be reduced to the following problem concerning the stress function: to determine a function $\psi(x, y)$ with continuous first derivatives which vanishes along the contour of the given cross section, satisfies $\psi_{xx}+\psi_{yy}=-c^2$ for a given c with $\psi_x^2+\psi_y^2<1$ in part of the cross section and satisfies $\psi_x^2+\psi_y^2=1$ in the rest. While Nadai's well-known soap film and sand hill analogy makes plausible the existence of the function $\psi(x, y)$, the present paper represents the first rigorous investigation of this existence problem. The generality of the investigation is restricted by the assumption that the elastic core nowhere extends to the contour of the cross section.

W. Prager (Providence, R. I.).

Stevenson, A. C. The centre of flexure of a hollow shaft. Proc. London Math. Soc. (2) **50**, 536–549 (1949).

This paper may be taken as supplementing two papers of B. R. Seth in which the bending of a hollow shaft with an eccentric hole is discussed [cf. Proc. Indian Acad. Sci., Sect. A. **4**, 531–541 (1936); **5**, 23–31 (1937)]. In a previous paper the author [Philos. Trans. Roy. Soc. London. Ser. A. **237**, 161–229 (1938)] defined the centre of flexure as a load point in the section of a cantilever beam such that the mean twist taken over the section is zero and obtained results for a large number of sections with the help of some canonical flexure functions and moment integrals. Using this method he now solves the flexure problem for a hollow shaft and obtains the position of the centre of flexure. When the circular boundaries of the shaft are in contact the results are expressed in terms of trigamma and tetragamma functions and numerical values are obtained for the associated twist and the centre of flexure.

B. R. Seth.

Takeyama, Hisao. On the torsional rigidity of rectangular prisms made of various materials. J. Soc. Appl. Mech. Japan **2**, 88–91, 110 (1949). (Japanese. English summary)

The torsional rigidity is calculated for a prismatic bar of rectangular cross section composed of layers having different elastic constants.

From the author's summary.

Giovannozzi, Renato. Trazione, torsione e flessione pura di solidi svergolati a sezione costante. Pont. Acad. Sci. Comment. **6**, 183–221 (1942).

Ōkubo, H. On the torsion of a prismatic cylinder with a star-shaped section. J. Appl. Phys. **20**, 1155–1157 (1949).

The torsion problem is treated for a prism with a star-shaped section when the boundaries of the serrated notches are considered as logarithmic spirals. Harmonic torsion functions in infinite series in θ are constructed with positive and negative powers of the radial variable r. One of these holds for $r<a$, the other for $r>a$, where a is the radius of the shaft at the bottom of the notch. To satisfy the boundary condition one is led to a complicated system of infinitely many equations. In one example of a shaft having 18 symmetrical notches, only three terms in the series are used and the torque is calculated. This result is confirmed by an experiment.

D. L. Holl (Ames, Iowa).

Ōkubo, H. Bending of a thin circular plate of an aeolotropic material under uniform lateral load (supported edge). J. Appl. Phys. **20**, 1151–1154 (1949).

The problem described in the title is solved using curvilinear coordinates. Since the final solution involves an infinite set of equations, the author suggests approximations and applies these successfully to the problem of the bending of an oak plate which is cut parallel to the grain. Empirical expressions for the deflected form and moments are then given, and the method of solution for a simply supported elliptic plate is indicated.

H. D. Conway.

Dzhanelidze, G. Yu. Survey of the work published in the USSR on the theory of the bending of thick and thin plates. Amer. Math. Soc. Translation no. **6**, 28 pp. (1950).

Translated from Akad. Nauk SSSR. Prikl. Mat. Meh. **12**, 109–128 (1948); these Rev. **9**, 481, 735.

Naruoka, Masao. On the calculation of a symmetrically deformed circular plate with variable thickness. J. Soc. Appl. Mech. Japan **2**, 87–88, 110 (1949). (Japanese. English summary)

In an attempt to simplify the calculations the author has adapted the slope deflection method to this problem and has obtained satisfactory results. *From the author's summary.*

Burić, Milan. Solution of the steady state problem for the rectangular plate using the orthogonal functions of the transversal oscillations of the beam. Glas Srpske Akad. Nauka **195**, 105–112 (1949). (Serbian)

Hlitčijev, J. On longitudinal ribs in a compressed plate. With an appendix by J. Karamata. Glas Srpske Akad. Nauka **195**, 17–36 (1949). (Serbian)

The problem of a plate reinforced by longitudinal ribs treated by Timoshenko [Theory of Elastic Stability, McGraw-Hill, New York-London, 1936, pp. 372 ff.]. Numerical results were given only for the case of two or three ribs. The author shows that the problem of optimal dimensions can be treated for an arbitrary number of ribs using closed analytic expressions which Karamata obtained for the series $\sum_{-\infty}^{+\infty}\{\lambda-(1+(2sn+t)^2)^2\}^{-1}$ in terms of trigonometric and hyperbolic functions. *W. Feller* (Ithaca, N. Y.).

Libove, Charles, and Batdorf, S. B. A general small-deflection theory for flat sandwich plates. Tech. Rep. Nat. Adv. Comm. Aeronaut., no. **899**, 18 pp. (1948).

Previously issued as Tech. Notes Nat. Adv. Comm. Aeronaut., no. **1526** (1948); these Rev. **10**, 86.

Reissner, E. Errata: Finite deflections of sandwich plates. J. Aeronaut. Sci. **17**, 125 (1950).

Cf. the same J. **15**, 435–440 (1948); these Rev. **10**, 273.

Stowell, Elbridge Z. Plastic buckling of a long flat plate under combined shear and longitudinal compression. Tech. Notes Nat. Adv. Comm. Aeronaut., no. **1990**, 17 pp. (1949).

The author's prior work on shear alone and compression alone [same Tech. Notes, nos. **1556**, **1681** (1948); these Rev. **10**, 82, 760] is extended to the title problem. Buckling is found to occur approximately when

$$R_c(E_s)_{pc}/(E_s)_{\sigma_i}+[R_s(E_s)_{ps}/(E_s)_{\sigma_i}]^2=1,$$

where R_c is the ratio of compressive stress when buckling occurs under the combined loading to the compressive stress when buckling occurs under compression alone and R_s the corresponding ratio for shear stress; E_s is the secant modulus, subscript pc indicates compression alone, ps shear alone and σ_i combined stress. A deformation theory of plasticity is used in the present and previous derivations.

D. C. Drucker (Providence, R. I.).

***Favre, Henry. The influence of its own weight on the stability of a rectangular plate.** Proc. Seventh Internat. Congress Appl. Mech., 1948, v. 1, pp. 151–159.

Kroll, Wilhelmina D. Instability in shear of simply supported square plates with reinforced hole. J. Research Nat. Bur. Standards **43**, 465–472 (1949).

Tolotti, Carlo. Sulla statica delle superficie inestendibili ed elasticamente flessibili. Giorn. Mat. Battaglini (4) **2**(**78**), 128–150 (1949).

In previous papers [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **1**, 187–192, 369–374, 605–609 (1946); these Rev. **8**, 359] the author has generalized Beltrami's differential equations for the deformation of flexible but inextensible membranes by introducing terms arising from flexural elasticity. He now formulates the principle of virtual work in a form applicable to a deformed surface. By using the Gauss-Codazzi equations he deduces his former differential equations and associated boundary conditions as well. *C. Truesdell* (Washington, D. C.).

Goldenveizer, A. L., and Lurye, A. I. The mathematical theory of the equilibrium of elastic shells. Amer. Math. Soc. Translation no. **9**, 54 pp. (1950).

Translated from Akad. Nauk SSSR. Prikl. Mat. Meh. **11**, 565–592 (1947); these Rev. **9**, 396.

Muštari, H. M. Nonlinear theory of the equilibrium of the boundary zone of an elastic shell. Doklady Akad. Nauk SSSR (N.S.) **69**, 511–513 (1949). (Russian)

In this sequel to the author's earlier paper [Akad. Nauk SSSR. Prikl. Mat. Meh. **13**, 121–134 (1949); these Rev. **11**, 69] the nonlinear equilibrium equations (2.4) and (2.5) of that paper are replaced by two approximate equations obtained by assuming $\epsilon h^{-1}\ll 1$, where ϵ is the maximum elongation and $2h$ is the ratio of the thickness of the shell to its smallest linear dimension. It is shown that the use of linearized equations in determining the edge effect leads to serious errors. *I. S. Sokolnikoff* (Los Angeles, Calif.).

Nazarov, A. A. On the theory of thin sloping shells. Akad. Nauk SSSR. Prikl. Mat. Meh. **13**, 547–550 (1949). (Russian)

The author writes out approximate equations of equilibrium (taking account of bending stresses) for a thin sloping shell in a "nearly Cartesian coordinate system," which is formed by intersecting the surface of the shell by two families of mutually orthogonal planes.

I. S. Sokolnikoff (Los Angeles, Calif.).

Grigoryan, D. M. A normal impact on an unbounded thin membrane. Akad. Nauk SSSR. Prikl. Mat. Meh. **13**, 277–284 (1949). (Russian)

The author studies the elastic and plastic deformations of a thin unbounded membrane under impact by the vertex of a right circular cone whose axis is moving with uniform velocity normal to the membrane. The equations of motion are integrated and the results are tabulated.

T. C. Doyle (Hanover, N. H.).

***Huber, M. T. The bending of the curved tube of elliptic section.** Proc. Seventh Internat. Congress Appl. Mech., 1948, v. 1, 322–328.

Uflyand, Ya. S. Exact solution of the problem of bending of a prismatic bar for a class of asymmetrical cross sections. Doklady Akad. Nauk SSSR (N.S.) **69**, 751–754 (1949). (Russian)

A solution is given of Saint Venant's flexure problem for a prism whose cross-section is formed by the arcs of two intersecting circles. The problem is solved in bipolar coordinates by use of Fourier integrals. The torsion problem was solved by the same technique in the same Doklady (N.S.) **68**, 17–20 (1949); these Rev. **11**, 288.

I. S. Sokolnikoff (Los Angeles, Calif.).